①

Vector Spaces That Are Not $\mathbb{R}^m$

1a) Suppose $\vec{0}$ and $\vec{0}'$ are add idents.

① $\vec{0}+\vec{0}' = \vec{0}$ B.c. $\vec{0}'$ is add ident.

② $\vec{0}'+0 = \vec{0}'$ B.c. $\vec{0}$ is add ident.

$\vec{0}+\vec{0}' = \vec{0}'+\vec{0}$ add commutes

$\overset{\shortparallel}{\vec{0}} \quad \overset{\shortparallel}{\vec{0}'}$ by ① & ②

$\therefore \boxed{\vec{0} = \vec{0}'}$

b) Let $\vec{X}'$ be $\vec{X}$'s add inverse

① $\vec{0} = \vec{X}+\vec{X}'$ by prop a-4

$= \vec{X}'+\vec{X}$ add commutes

$= X'+1\vec{X}$ by prop m-2

$= X'+(1+0)\vec{X}$ Scalars $1=1+0$

$= X'+(1\vec{X}+0\vec{X})$ by d-2

$= \vec{X}'+(\vec{X}+0\vec{X})$ by m-2

$= (\vec{X}'+\vec{X})+0\vec{X}$ by a-1

② $= 0+0\vec{X} = 0\vec{X}$ by a-4

Lines ① and ② say $\vec{0} = 0\vec{X}$ for any vector $\vec{X}$.

(2)

c) Suppose $\vec{X}'$ & $\vec{X}''$ are add inverses for $\vec{X}$

(1) $(\vec{X}+\vec{X}')+\vec{X}'' = \vec{O}+\vec{X}'' = \vec{X}''$ by a-4

$\parallel$

$X+(X'+X'')$ by add is assoc
and add commutes

$= X+(X''+X')$

$= (X+X'')+X'$ assoc property

$= O+\vec{X}'$ b/c X'' is add inverse

(2) $= \vec{X}'$ $\therefore$ $\boxed{\text{Lines (1) \& (2) combine to show } X'=X''}$

d) Let $\vec{X}'$ be add inverse for $\vec{X}$.

(1) $X' = X'+\vec{O} = \vec{X}'+0\vec{X}$ by part b

$= X'+(1+-1)\vec{X}$ scalar $0=1+(-1)$

$= (X'+1\vec{X})+(-1)\vec{X}$ by d-2

$= (X'+\vec{X})+(-1)\vec{X}$ by m-2

$= \vec{O}+(-1)\vec{X}$ X' is X's add inv

(2) $= (-1)\vec{X}$ by a-3

$\boxed{\therefore \text{ (1) \& (2) combine to show } X'=(-1)\vec{X}}$

(3)

2a) Let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$; $\vec{0} = \begin{pmatrix} o_1 \\ o_2 \end{pmatrix}$

$$\vec{x} + \vec{0} = \vec{x}$$

$$\Rightarrow \begin{pmatrix} x_1 + o_1 - 1 \\ x_2 + o_2 - 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{matrix} x_1 + o_1 - 1 = x_1 \\ x_2 + o_2 - 1 = x_2 \end{matrix}$$

Let $x' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$

$$\begin{matrix} o_1 = 1 \\ o_2 = 1 \end{matrix} \Rightarrow \boxed{\vec{0} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

$$\vec{x} + \vec{x}' = \vec{0}$$

$$\Rightarrow \begin{pmatrix} x_1 + x_1' - 1 \\ x_2 + x_2' - 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \begin{matrix} x_1 + x_1' - 1 = 1 \\ x_2 + x_2' - 1 = 1 \end{matrix}$$

$$\begin{matrix} x_1' = 2 - x_1 \\ x_2' = 2 - x_2 \end{matrix} \Rightarrow \boxed{\vec{x}' = \begin{pmatrix} 2 - x_1 \\ 2 - x_2 \end{pmatrix}}$$

b) $(\alpha + \beta)\vec{x} = \begin{pmatrix} (\alpha+\beta)(x_1 - 1) + 1 \\ (\alpha+\beta)(x_2 - 1) + 1 \end{pmatrix}$

$$= \begin{pmatrix} \alpha(x_1 - 1) + \beta(x_1 - 1) + 1 \\ \alpha(x_2 - 1) + \beta(x_2 - 1) + 1 \end{pmatrix} = \text{(over)}$$

④

$$= \begin{pmatrix} \alpha(x_1-1)+1 & + \beta(x_1-1)+1 & -1 \\ \alpha(x_2-1)+1 & + \beta(x_2-1)+1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha(x_1-1)+1 \\ \alpha(x_2-1)+1 \end{pmatrix} + \begin{pmatrix} \beta(x_1-1)+1 \\ \beta(x_2-1)+1 \end{pmatrix}$$

$$= \alpha \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \beta \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \alpha\vec{x} + \beta\vec{x}$$

d) $0\vec{x} = \begin{pmatrix} 0(x_1-1)+1 \\ 0(x_2-1)+1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \vec{0}$ ✓

see part a

$-1\vec{x} = \begin{pmatrix} -1(x_1-1)+1 \\ -1(x_2-1)+1 \end{pmatrix} = \begin{pmatrix} 2-x_1 \\ 2-x_2 \end{pmatrix} = -\vec{x}$ ✓

3) $\vec{0} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\vec{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\vec{y} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

Suppose $\alpha \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \vec{0} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(area)

(5)

$$= \begin{pmatrix} \alpha(2-1)+1 \\ \alpha(1-1)+1 \end{pmatrix} + \begin{pmatrix} \beta(4-1)+1 \\ \beta(2-1)+1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha(2-1)+1 + \beta(4-1)+1 \quad -1 \\ \alpha(1-1)+1 + \beta(2-1)+1 \quad -1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha+3\beta+1 \\ \beta+1 \end{pmatrix} \Rightarrow \begin{matrix} \alpha+3\beta+1 = 1 \\ \beta+1 = 1 \end{matrix}$$

$$\Downarrow$$

$$\begin{matrix} \alpha+3\beta = 0 \\ \beta = 0 \end{matrix} \Rightarrow \alpha = 0$$

So $\alpha = \beta = 0$. $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ & $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ are indep

b) $\text{Span}\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \end{pmatrix}\right\}$

$$= \left\{\alpha\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \beta\begin{pmatrix} 4 \\ 2 \end{pmatrix} : \alpha \in \mathbb{R}, \beta \in \mathbb{R}\right\}$$

$$= \left\{\begin{pmatrix} \alpha+3\beta+1 \\ \beta+1 \end{pmatrix} : \alpha \in \mathbb{R}, \beta \in \mathbb{R}\right\}$$

⑥

So for any $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in S^2$ need

α & β such that

$$\begin{cases} \alpha + 3\beta + 1 = z_1 \\ \beta + 1 = z_2 \end{cases} \Rightarrow \beta_2 = z_2 - 1$$

& $\alpha = z_1 - 3(z_2 - 1) - 1$

$= z_1 - 3z_2 + 2$

So $(z_1 - 3z_2 + 2)\begin{pmatrix} 2 \\ 1 \end{pmatrix} + (z_2 - 1)\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$

4) $\vec{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\vec{y} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$$\alpha\vec{x} + \beta\vec{y} = \begin{pmatrix} \alpha(2-1)+1 \\ \alpha(1-1)+1 \end{pmatrix} + \begin{pmatrix} \beta(3-1)+1 \\ \beta(1-1)+1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha+1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2\beta+1 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha+1+2\beta+1-1 \\ 1+1-1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha+2\beta+1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \vec{0} \quad (\text{zero})$$

⑦

$\Rightarrow \alpha + 2\beta + 1 = 1$ take $\beta = 1$

$\alpha = -2$

So

$$-2\begin{pmatrix}2\\1\end{pmatrix} + \begin{pmatrix}3\\1\end{pmatrix} = \vec{0}$$ Not both zero

$$2\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix} \qquad (2\vec{x} = \vec{y})$$

These are not independent.

b) $\text{span}\{\vec{x}, \vec{y}\} = \left\{\alpha\begin{pmatrix}2\\1\end{pmatrix} + \beta\begin{pmatrix}3\\1\end{pmatrix} : \begin{matrix}\alpha \in \mathbb{R}\\ \beta \in \mathbb{R}\end{matrix}\right\}$

$$= \left\{\begin{pmatrix}\alpha + 2\beta + 1\\ 1\end{pmatrix} : \alpha \in \mathbb{R}, \beta \in \mathbb{R}\right\}$$

$$= \left\{\begin{pmatrix}(\alpha + 2\beta)(2-1) + 1\\ (\alpha + 2\beta)(1-1) + 1\end{pmatrix} : \alpha \in \mathbb{R}, \beta \in \mathbb{R}\right\}$$

$$= \left\{(\alpha + 2\beta)\begin{pmatrix}2\\1\end{pmatrix} : \alpha, \beta \in \mathbb{R}\right\}$$

$\alpha + 2\beta = \tilde{\alpha} \in \mathbb{R}$

$$= \left\{\tilde{\alpha}\begin{pmatrix}2\\1\end{pmatrix} : \tilde{\alpha} \in \mathbb{R}\right\}$$

5) From calc notice for any $k < m$

$$\frac{d^k}{dx^k} x^m = m(m-1)\cdots(m-k+1)\, x^{m-k}$$

For any $k > m$

$$\frac{d^k}{dx^k} x^m = 0$$

and when $k = m$

$$\frac{d^k}{dx^k} x^m = m!$$

Suppose

$$\alpha_0 \cdot 1 + \alpha_1 x + \cdots + \alpha_n x^n = 0 \quad \forall x$$

take the kth der to get $(k \leq n)$

$$0 = \alpha_k k! + \alpha_{k+1}(k+1)\cdots 2\, x^1 + \alpha_{k+2}(k+2)\cdots 3\, x^2 + \cdots + \alpha_n n(n-1)\cdots(n-k+1)\, x^{n-k}$$

Set $x = 0$ to get

$$0 = k!\,\alpha_k \Rightarrow \alpha_k = 0 \quad \forall k < n$$

⑨

also

$$\frac{d^n}{dx^n}(\alpha_0 + \cdots + \alpha_n x^n) = 0$$

$$\alpha_n n! \Rightarrow \alpha_n = 0$$

i.e. $\alpha_0 = \alpha_1 = \cdots = \alpha_n = 0$

$\Rightarrow \{1, x, \cdots x^n\}$ is linearly independent.

b) Suppose

a) $\alpha_0 (x-x_1)(x-x_2) + \alpha_1 (x-x_0)(x-x_2) + \alpha_2 (x-x_0)(x-x_1) = 0 \quad \forall x$

Let $x = x_0$ to get

$\alpha_0 (x_0-x_1)(x_0-x_2) + \alpha_1 \cdot 0 + \alpha_2 \cdot 0 = 0$

$\Rightarrow \alpha_0 = 0$

Note $(x_0-x_1)(x_0-x_2) \neq 0$

Let $x = x_1$ to get

$\alpha_0 \cdot 0 + \alpha_1 (x_1-x_0)(x_1-x_2) + \alpha_2 \cdot 0 = 0$

$\Rightarrow \alpha_1 = 0$

Note $(x_1-x_0)(x_1-x_2) \neq 0$

(over)

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Note $(x_2-x_0)(x_2-x_1) \neq 0$

Let $x = x_2$ to get

$\alpha_0 \cdot 0 + \alpha_1 \cdot 0 + \alpha_2 (x_2 - x_0)(x_2 - x_1) = 0$

$\Rightarrow \alpha_2 = 0$

$\therefore \{(x-x_1)(x-x_2), (x-x_0)(x-x_2), (x-x_0)(x-x_1)\}$

forms an independent set.

b) Suppose

$P(x) = \alpha_0 (x-x_1)(x-x_2) + \alpha_1 (x-x_0)(x-x_2) + \alpha_2 (x-x_0)(x-x_1)$

Let $x = x_0$

$P(x_0) = \alpha_0 (x_0 - x_1)(x_0 - x_2) + 0 + 0$

$\Rightarrow \alpha_0 = P(x_0) / (x_0 - x_1)(x_0 - x_2)$

Let $x = x_1$

$P(x_1) = 0 + \alpha_1 (x_1 - x_0)(x_1 - x_2) + 0$

$\Rightarrow \alpha_1 = P(x_1) / (x_1 - x_0)(x_1 - x_2)$

Let $x = x_2$

$P(x_2) = 0 + 0 + \alpha_2 (x_2 - x_0)(x_2 - x_1)$

$\Rightarrow \alpha_2 = P(x_2) / (x_2 - x_0)(x_2 - x_1)$

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Suppose

7a) $\alpha_0 1 + \alpha_1(x-0) + \alpha_2(x-0)(x-1) = 0$

$\forall x$

Let $x=0$

$\alpha_0 + 0 = 0 \Rightarrow \alpha_0 = 0$

Let $x=1$

$\alpha_0 + \alpha_1 + 0 = 0$

But since $\alpha_0 = 0 \Rightarrow \alpha_1 = 0$

Let $x=2$

$\alpha_0 + 2\alpha_1 + \alpha_2 = 0$

But since $\alpha_1 = \alpha_0 = 0 \Rightarrow \alpha_2 = 0$

So $\{1, (x-0), (x-0)(x-1)\}$ is Linearly independent

$\forall x$

b) $\alpha_0(2x+1) + \alpha_1(x^2) + \alpha_2(x+1)^2 = 0$

Let's expand this into the standard basis $\{1, x, x^2\}$.

$0 = \alpha_0(2x+1) + \alpha_1 x^2 + \alpha_2(x^2+2x+1)$

$= (\alpha_0 + \alpha_2)1 + (2\alpha_0 + 2\alpha_2)x + (\alpha_1 + \alpha_2)x^2$

(over)

But This implies

$$\alpha_0 + \alpha_2 = 0$$

$$2(\alpha_0 + \alpha_2) = 0 \Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(\alpha_1 + \alpha_2) = 0 \qquad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

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$$\begin{aligned} \alpha_2 &= \alpha \\ \alpha_1 &= -\alpha \\ \alpha_0 &= -\alpha \end{aligned} \Leftarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

WLOG take $\alpha = 1$. So we see

$$-1(2x+1) - 1x^2 + 1(x+1)^2$$

$$= \cancel{-2x} \cancel{-1} \cancel{-x^2} + \cancel{x^2} + \cancel{2x} + \cancel{1} = 0 \quad \forall x$$

$\therefore \{(2x+1), x^2, (x+1)^2\}$ is a **dependent** set.

b) Consider $\{x-1, x+1, x^2+2\}$

(over)

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a) $1 = \alpha_0(X-1) + \alpha_1(X+1) + \alpha_2(X^2+2)$

Expand into the standard basis $\{1, X, X^2\}$

$1 = (-\alpha_0 + \alpha_1 + 2\alpha_2)1 + (\alpha_0 + \alpha_1)X + \alpha_2 X^2$

$$\Leftrightarrow \begin{array}{l} \alpha_2 = 0 \\ \alpha_0 + \alpha_1 = 0 \\ -\alpha_0 + \alpha_1 + 2\alpha_2 = 1 \end{array} \Rightarrow \begin{array}{l} \alpha_0 = -1/2 \\ \alpha_1 = 1/2 \\ \alpha_2 = 0 \end{array}$$

so

$$\boxed{1 = -1/2(X-1) + 1/2(X+1) + 0(X^2+2)}$$

b) $X = \alpha_0(X-1) + \alpha_1(X+1) + \alpha_2(X^2+2)$

$= (-\alpha_0 + \alpha_1 + 2\alpha_2)1 + (\alpha_0 + \alpha_1)X + \alpha_2 X^2$

$$\Leftrightarrow \begin{array}{l} \alpha_2 = 0 \\ \alpha_0 + \alpha_1 = 1 \\ -\alpha_0 + \alpha_1 + 2\alpha_2 = 0 \end{array} \Rightarrow \begin{array}{l} \alpha_0 = 1/2 \\ \alpha_1 = 1/2 \\ \alpha_2 = 0 \end{array}$$

so

$$\boxed{X = 1/2(X-1) + 1/2(X+1) + 0(X^2+2)}$$

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c) $x^2 = \alpha_0(x-1) + \alpha_1(x+1) + \alpha_2(x^2+2)$

$= (-\alpha_0 + \alpha_1 + 2\alpha_2)\cdot 1 + (\alpha_0 + \alpha_1)x + \alpha_2 x^2$

$\Rightarrow$ $\alpha_2 = 1$ $\qquad\qquad$ $\alpha_0 = 1$

$\alpha_0 + \alpha_1 = 0$ $\quad \Rightarrow \quad$ $\alpha_1 = -1$

$-\alpha_0 + \alpha_1 + 2\alpha_2 = 0$ $\qquad$ $\alpha_2 = 1$

So

$$x^2 = 1(x-1) - 1(x+1) + 1(x^2+2)$$

d) Use parts (a) (b) & (c) on this

$x^2 + 3x + 4$

$= \left[1(x-1) - 1(x+1) + 1(x^2+2)\right]$ part c

$+ 3\left[\tfrac{1}{2}(x-1) + \tfrac{1}{2}(x+1) + 0(x^2+2)\right]$ part b

$+ 4\left[-\tfrac{1}{2}(x-1) + \tfrac{1}{2}(x+1) + 0(x^2+2)\right]$ part a

$= (1 + 3/2 - 4/2)(x-1) + (-1 + 3/2 + 4/2)(x+1)$

$+ (1 + 0 + 0)(x^2+2)$