① Gram-Schmidt & Shur's Lemma

1) $e_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Use $x_2 \notin span\{e_1\}$ $x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$e_2 = x_2 - \alpha_1 e_1$ where $\alpha_1 = \frac{x_1 \cdot e_1}{e_1 \cdot e_1} = \frac{1}{3}$,

$= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 2/3 \\ -1/3 \end{pmatrix}$

I'll renormalize to $e_2 = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$

Use $x_3 \notin span\{e_1, e_2\}$ $x_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$e_3 = x_3 - \alpha_1 e_1 - \alpha_2 e_2$

where $\alpha_1 = \frac{x_3 \cdot e_1}{e_1 \cdot e_1} = \frac{1}{3}$

$\alpha_2 = \frac{x_3 \cdot e_2}{e_2 \cdot e_2} = \frac{-1}{6}$

$e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$ = over

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$$= \begin{pmatrix} 0 + 1/3 - 1/6 \\ 0 - 1/3 + 2/6 \\ 1 - 1/3 - 1/6 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix}.$$

I'll renormalize e_3 to $e_3' = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

So

$$\boxed{\{e_1, e_2, e_3\}} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

2) $e_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ $x_1 \notin \text{span}\{e_1\}$ take $x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$e_2 = x_2 - \alpha_1 e_1$ where $\alpha_1 = \frac{x_1 \cdot e_1}{e_1 \cdot e_1} = 0$

So $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$x_3 \notin \text{span}\{e_1, e_2\}$ take $x_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$e_3 = x_3 - \alpha_1 e_1 - \alpha_2 e_2$ where

$\alpha_1 = \frac{x_3 \cdot e_1}{e_1 \cdot e_1} = \frac{1}{2}$ $\alpha_2 = \frac{x_3 \cdot e_2}{e_2 \cdot e_2} = 0$

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So $e_3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} = \begin{pmatrix}-1/2\\0\\1/2\\0\end{pmatrix}$

I'll renormalize e_3 to $e_3 = \begin{pmatrix}-1\\0\\1\\0\end{pmatrix}$

Need $x_4 \notin \text{span}\{e_1, e_2, e_3\}$ take $x_4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$,

$e_4 = x_4 - \alpha_1 e_1 - \alpha_2 e_2 - \alpha_3 e_3$ where

$\alpha_1 = \frac{x_4 \cdot e_1}{e_1 \cdot e_1} = 0$, $\alpha_2 = \frac{x_4 \cdot e_2}{e_2 \cdot e_2} = 0$

$\alpha_3 = \frac{x_4 \cdot e_3}{e_3 \cdot e_3} = 0$

So $e_4 = x_4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

$$\boxed{\{e_1, e_2, e_3, e_4\} = \left\{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}}$$

3) $e_1 = \begin{pmatrix}1\\3\end{pmatrix}$, $e_2 = \begin{pmatrix}3\\-1\end{pmatrix}$, $\hat{e}_1 = \frac{1}{\sqrt{10}}\begin{pmatrix}1\\3\end{pmatrix}$

$\|e_1\| = \sqrt{1+9} = \sqrt{10} \Rightarrow$

$\|e_2\| = \sqrt{9+1} = \sqrt{10}$ $\hat{e}_2 = \frac{1}{\sqrt{10}}\begin{pmatrix}3\\-1\end{pmatrix}$

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④

So $\boxed{O_3 = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 & 3 \\ 3 & -1 \end{pmatrix}}$

Check $O_3^T O_3 = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 & 3 \\ 3 & -1 \end{pmatrix} \cdot \frac{1}{\sqrt{10}}\begin{pmatrix} 1 & 3 \\ 3 & -1 \end{pmatrix}$

$= \frac{1}{10}\begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

4) $e_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $e_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

$\|e_1\| = \sqrt{4+9} = \sqrt{13}$ $\Rightarrow$ $\hat{e}_1 = \frac{1}{\sqrt{13}}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\|e_2\| = \sqrt{9+4} = \sqrt{13}$ $\hat{e}_2 = \frac{1}{\sqrt{13}}\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

So $\boxed{O_4 = \frac{1}{\sqrt{13}}\begin{pmatrix} 2 & 3 \\ 3 & -2 \end{pmatrix}}$

Check $O_4^T O_4 = \frac{1}{\sqrt{13}}\begin{pmatrix} 2 & 3 \\ 3 & -2 \end{pmatrix} \frac{1}{\sqrt{13}}\begin{pmatrix} 2 & 3 \\ 3 & -2 \end{pmatrix}$

$= \frac{1}{13}\begin{pmatrix} 13 & 0 \\ 0 & 13 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

(5)

5) Let $O = O_3 O_4$

$$= \frac{1}{\sqrt{10}}\begin{pmatrix} 1 & 3 \\ 3 & -1 \end{pmatrix} \frac{1}{\sqrt{13}}\begin{pmatrix} 2 & 3 \\ 3 & -2 \end{pmatrix}$$

$11^2 =$
11

$$\boxed{O = \frac{1}{\sqrt{130}}\begin{pmatrix} 11 & -3 \\ 3 & 11 \end{pmatrix}}$$

Check $O^T O = \frac{1}{130}\begin{pmatrix} 11 & 3 \\ -3 & 11 \end{pmatrix}\begin{pmatrix} 11 & -3 \\ 3 & 11 \end{pmatrix}$

$$= \frac{1}{130}\begin{pmatrix} 121+9 & 0 \\ 0 & 9+121 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

6) $M = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ Columns are orthogonal but don't have unit length.

$M^T M$ is diagonal but not I.

$$M^T M = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \checkmark$$ diagonal but not I.

(6)

FYI. Check that $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix}$

is an orthonormal matrix.

7) $A = \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix}$

Need an eigenvalue & eigenvector

$\det(A - \lambda I) = \det\begin{pmatrix} -1-\lambda & 2 \\ -3 & 4-\lambda \end{pmatrix} = (\lambda+1)(\lambda-4) + 6$

$= \lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2) = 0$

Let's use $\boxed{\lambda = 1}$

$[A - 1I] = \begin{bmatrix} -2 & 2 \\ -3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow \boxed{\overset{e_1}{r} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

Gram-Schmidt to get e_2. take $X_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$e_2 = X_2 - \alpha e_1$ where $\alpha = \frac{X_2 \cdot e_1}{e_1 \cdot e_1} = \frac{1}{2}$

$e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and renormalize

so $\{e_1, e_2\} = \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$

⑦

So for Schur we'll use $O = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

and so

$$O^T A O = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 1 & 7 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 10 \\ 0 & 4 \end{pmatrix}$$

So $\boxed{T = \begin{pmatrix} 1 & 5 \\ 0 & 2 \end{pmatrix}}$

Notice didn't get same answer as before. Before I used

$$O = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

8) $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

Need an eigenvalue and eigenvector

$$\det(A - \lambda I) = \det\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = (\lambda-2)^2 - 1 = 0$$

$$\lambda = 2 \pm 1$$

Let's use $\lambda = 1$.

For r

$$\left[A - 1I\right] = \begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

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$\Rightarrow r = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. So $e_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Gram-Schmidt with $x_2 \notin \text{span}\{e_1\}$

equal to say $x_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$e_2 = x_2 - \alpha e_1$ where $\alpha = \frac{x_2 \cdot e_1}{e_1 \cdot e_1} = \frac{1}{2}$

$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$ renormalize to $e_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

So

$\{e_1, e_2\} = \left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$

Normalize these to unit length to get

$$O = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$O^T A O = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 1 & 3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix} =$$ (over)

⑨

So $O^T A O = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ ← The other eigenvalue.

Notice This is diagonal and that has no gradient. This happened because A is <u>symmetric</u> for this problem, $A^T = A$,

9) $A = \begin{pmatrix} -1 & 6 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Need an eigenvalue & eigenvector

$$\det(A - \lambda I) = \det \begin{pmatrix} -1-\lambda & 6 & 0 \\ -1 & 4-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix}$$

$$= (1-\lambda) \det \begin{pmatrix} -1-\lambda & 6 \\ -1 & 4-\lambda \end{pmatrix} = 0$$

This tells us $\lambda = 1$ is an eigenvalue.
~~For~~ eigenvector do the following

$$[A - 1I] = \begin{bmatrix} -2 & 6 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \text{over}$$

(10)

$$\sim \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow$$ 2 dimensional eigenspace

Free Free

$r_2 = \alpha$ $r_3 = \beta$

$r_1 = 3r_2 = 3\alpha$

$$r = \alpha \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

2 independent eigenvecs

$$r = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

I'm going to use

$$r = e_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

I can read off $e_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ & $e_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

So $$O_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$O_1^T A O_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 6 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 6 \\ 0 & -1 & 4 \\ 1 & 0 & 0 \end{pmatrix} \boxed{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 6 \\ 0 & -1 & 4 \end{pmatrix}}$$

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(11)

I want to show you here how to do "block" matrix multiplication.

Suppose A & B are written in bock form:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

where $A_{11}, B_{11} \in \mathbb{R}^{k\times k}$ $A_{12}, B_{12} \in \mathbb{R}^{k\times l}$

$A_{21}, B_{21} \in \mathbb{R}^{l\times k}$ $A_{22}, B_{22} \in \mathbb{R}^{l\times l}$

(and so A and $B \in \mathbb{R}^{(k+l)\times(k+l)}$)

You should check that this is correct.

Then

$$AB = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

The order of mult matters here.

$$= \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$$

$$= \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = C$$

②

Here's how to use block matrix multiplication in Schur's lemma.

Suppose $A = \left(\begin{array}{c|c} \lambda & S \\ \hline Z & \tilde{A} \end{array}\right)$ $\quad S = (s_1 \cdots s_{n-1})$

$$Z = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{array}{l} (n-1) \\ \text{times} \end{array}$$

and Let $O = \left(\begin{array}{c|c} 1 & Z^T \\ \hline Z & \tilde{O} \end{array}\right)$ $\quad \tilde{A} \in \mathbb{R}^{(n-1)\times(n-1)}$

$\tilde{O} \in \mathbb{R}^{(n-1)\times(n-1)}$

$$\Rightarrow O^T = \left(\begin{array}{c|c} 1 & Z^T \\ \hline Z & \tilde{O}^T \end{array}\right)$$

So $$O^T A O = \left(\begin{array}{c|c} 1 & Z^T \\ \hline Z & \tilde{O}^T \end{array}\right)\left(\begin{array}{c|c} \lambda & S \\ \hline Z & \tilde{A} \end{array}\right)\left(\begin{array}{c|c} 1 & Z^T \\ \hline Z & \tilde{O} \end{array}\right)$$

$$= \left(\begin{array}{c|c} 1 & Z^T \\ \hline Z & \tilde{O}^T \end{array}\right)\left(\begin{array}{c|c} \lambda + \cancelto{0}{SZ} & \cancelto{0}{\lambda Z^T} + S\tilde{O} \\ \hline \cancelto{0}{Z} + \cancelto{0}{\tilde{A}Z} & \cancelto{0}{ZZ^T} + \tilde{A}\tilde{O} \end{array}\right)$$

This is a column vector of zeros

$$= \left(\begin{array}{c|c} 1 & Z^T \\ \hline Z & \tilde{O}^T \end{array}\right)\left(\begin{array}{c|c} \lambda & S\tilde{O} \\ \hline Z & \tilde{A}\tilde{O} \end{array}\right) = \left(\begin{array}{c|c} \lambda + \cancelto{0}{Z^T Z} & S\tilde{O} + \cancelto{0}{Z^T\tilde{A}\tilde{O}} \\ \hline \cancelto{0}{Z\lambda} + \cancelto{0}{\tilde{O}^T Z} & \cancelto{0}{ZS\tilde{O}} + \tilde{O}^T\tilde{A}\tilde{O} \end{array}\right)$$

$$= \left(\begin{array}{c|c} \lambda & S\tilde{O} \\ \hline Z & \tilde{O}^T\tilde{A}\tilde{O} \end{array}\right)$$

This is a row vector

$(**)$

Also notice That

$$O^T O = \left(\begin{array}{c|c} 1 & z^T \\ \hline z & \tilde{O}^T \end{array}\right)\left(\begin{array}{c|c} 1 & z^T \\ \hline z & \tilde{O} \end{array}\right)$$

$$= \left(\begin{array}{c|c} 1 & z^T \\ \hline z & \tilde{O}^T \tilde{O} \end{array}\right)$$ and since, $\tilde{O}^T \tilde{O} = I$

Then

$$O^T O = \left(\begin{array}{c|c} 1 & z^T \\ \hline z & I \end{array}\right) = I$$

Now, return to Bottom of page 10.

$$O_1^T A O_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 6 \\ 0 & -1 & 4 \end{pmatrix}$$

λ

s

$\tilde{A}$

This is in **Block** form where

$$z = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad s = (0, 0) \quad \tilde{A} = \begin{pmatrix} -1 & 6 \\ -1 & 4 \end{pmatrix}$$

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(14)

So for the 2x2 matrix $\tilde{A} = \begin{pmatrix} -1 & 6 \\ -1 & 4 \end{pmatrix}$

Let's figure out its eigenvalue

$\det(\tilde{A} - \lambda I) = \det \begin{pmatrix} -1-\lambda & 6 \\ -1 & 4-\lambda \end{pmatrix}$

$= (\lambda+1)(\lambda-4) + 6 = (\lambda-1)(\lambda-2) = 0$

So for eigenvalue $\lambda = 1$ Let's get its eigenvector

$\lambda = 1 \quad \left[\tilde{A} - 1 I\right] = \begin{bmatrix} -2 & 6 \\ -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow r = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and Let's take this as $\vec{e}_1$

Do Gram-Schmidt to get $\vec{e}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

Normalize $\{e_1, e_2\} = \left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \end{pmatrix}\right\}$ to

get $\tilde{O} = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 & 1 \\ 1 & -3 \end{pmatrix}$

15

Mult out

$$\tilde{O}^T\tilde{A}\tilde{O} = \frac{1}{\sqrt{10}}\begin{pmatrix}3 & 1\\ 1 & -3\end{pmatrix}\begin{pmatrix}-1 & 6\\ -1 & 4\end{pmatrix}\frac{1}{\sqrt{10}}\begin{pmatrix}3 & 1\\ 1 & -3\end{pmatrix}$$

$$= \frac{1}{10}\begin{pmatrix}10 & -70\\ 0 & 20\end{pmatrix} = \begin{pmatrix}1 & -7\\ 0 & 2\end{pmatrix} = T_{2\times 2}$$

So the 3x3 T is

see * on page 10 or 13.

$\lambda = 1$

$$T = \begin{pmatrix}1 & * & *\\ 0 & 1 & -7\\ 0 & 0 & 2\end{pmatrix}$$

$T_{2\times 2}$

where $(*\ *) = \underbrace{(0\ 0)}\tilde{O} = (0\ 0)$

So $\tilde{O}$ on ** p 12

So $T = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & -7\\ 0 & 0 & 2\end{pmatrix}$

and $O_2 = \begin{pmatrix}1 & 0 & 0\\ 0 & \frac{1}{\sqrt{10}}\begin{pmatrix}3 & 1\\ 1 & -3\end{pmatrix} & \\ 0 & & \end{pmatrix}$

Notice the -7 in this T is different from my earlier answer. No surprise, I used a diffrnt O.

$$(O_1O_2)^T A O_1O_2 = O^TAO = T = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & -7\\ 0 & 0 & 2\end{pmatrix}$$