1

3331 HW 3

1) $\left(\frac{ds}{dt}\right)_{in} = 1$ units are in lbs/min

$\left(\frac{ds}{dt}\right)_{out} = -\frac{s}{20}$

$\frac{ds}{dt} = 1 - \frac{s}{20}$ with $s(0) = 0$

$\frac{ds}{dt} + \frac{1}{20} s = 1$

$\Rightarrow s(t) = ce^{-t/20} + 20$

$0 = s(0) = c + 20 \Rightarrow c = -20$

so $s(t) = -20e^{-t/20} + 20$

$Vol(t) = 20 + t$ gals

2a) $\left(\frac{ds}{dt}\right)_{in} = 2$

b) $\left(\frac{ds}{dt}\right)_{out} = \frac{s(t)}{20+t} \frac{\text{lbs}}{\text{gals}} \cdot \frac{1 \text{ gal}}{\text{min}} = \frac{s}{20+t} \frac{\text{lbs}}{\text{min}}$

c) $\frac{ds}{dt} = 2 - \frac{s}{20+t}$ with $s(0) = 0$

—over—

(2)

$$\frac{ds}{dt} + \frac{1}{20+t} s = 2$$

$$a(t) = \frac{1}{20+t} \qquad A(t) = \log(20+t)$$

$$e^{-A(t)} \frac{d}{dt}\left(e^{A(t)} s\right) = 2$$

$$\int_0^t \frac{d}{dt}\left((20+t)s\right) = 2\int_0^t (20+t)$$

$$= 40t + t^2$$

$$(20+t)s(t) - (20+0)s(0)$$

$$0$$

So $$\boxed{s(t) = \frac{40t+t^2}{20+t}}$$

3a) $$\frac{dz}{dt} = -\underbrace{\sqrt{2g}\frac{A_h}{A_T}}_{=k}\sqrt{z} \quad \text{with } z(0) = H$$

$$\int_{z=H}^{z(T)} \frac{dz}{\sqrt{z}} = -k\int_0^T dt = -kT$$

$$2\sqrt{z}\Big|_H^z = -kT$$

over

③

$$\sqrt{z} - \sqrt{H} = -\frac{k}{2} T$$

$$z(T) = 0 \Rightarrow \sqrt{H} = \frac{k}{2} T$$

$$\Rightarrow T = \frac{2\sqrt{H}}{k}$$

b) Suppose $\left(\frac{dV}{dt}\right)_{in} = I$

$$\left(\frac{dV}{dt}\right)_{out} = A_h \sqrt{2g}\, \sqrt{z}$$

$$A_T \frac{dz}{dt} = \frac{dV}{dt} = \left(\frac{dV}{dt}\right)_{in} - \left(\frac{dV}{dt}\right)_{out}$$

This gives

$$\frac{dz}{dt} = \frac{I}{A_T} - \overbrace{\frac{A_h\sqrt{2g}}{A_T}}^{=k} \sqrt{z}$$

with $z(0) = H$

$$\frac{dz}{dt} = \frac{I}{A_T} - k\sqrt{z} = \frac{I}{A_T}\left(1 - \frac{k A_T}{I}\sqrt{z}\right)$$

④

Let $\left(\frac{kA_T}{I}\right)^2 z = \tilde{z}$

so

$$\left(\frac{kA_T}{I}\right)^2 \frac{dz}{dt} = \underbrace{\left(\frac{kA_T}{I}\right)^2 \frac{I}{A_T}}_{\tilde{k} > 0}\left(1 - \frac{kA_T}{I}\sqrt{z}\right)$$

$$\begin{cases} \frac{d\tilde{z}}{dt} = \tilde{k}\left(1 - \sqrt{\tilde{z}}\right) \leftarrow \text{ODE} \\ \tilde{z}(0) = \left(\frac{kA_T}{I}\right)^2 H > 0 \end{cases}$$

Case 1 Now, if $\tilde{z}(0) < 1$ the ODE says $1 - \sqrt{\tilde{z}} > 0 \Rightarrow \frac{d\tilde{z}}{dt} > 0$

In this case $\tilde{z}(t)$ will increase until $\tilde{z}(t) = 1$ (if ever gets there)

Case 2 Now, if $\tilde{z}(0) > 1$ the ODE says $1 - \sqrt{\tilde{z}} < 0 \Rightarrow \frac{d\tilde{z}}{dt} < 0$

and in this case $\tilde{z}(t)$ will decrease until $\tilde{z}(t) = 1$ (if ever gets there)

(5)

Case 3 Finally, if $\tilde{z}(0) = 1$ This says

$$1 - \sqrt{\tilde{z}} = 0 \Rightarrow \frac{d\tilde{z}}{dt} = 0.$$

This says $\tilde{z}(t) = 1 \quad \forall t$.

In either Case 1 or Case 2 $\tilde{z}(t)$ can never get to $\tilde{z}(t^*) = 1$ in finite time. To see this let's integrate

$$\int_{\tilde{z}(0)}^{\tilde{z}(t)} \frac{d\tilde{z}}{1 - \sqrt{\tilde{z}}} = \int_0^{t^*} k\, dt$$

make $W = 1 - \sqrt{\tilde{z}}$ substitution.

$$\| $$

$$\underbrace{2\left[(1 - \sqrt{\tilde{z}}) - \log(1 - \sqrt{\tilde{z}})\right]}_{\text{This will go to } \infty \text{ if } \tilde{z} \to 1} = \tilde{k}t + 2\left[(1 - \sqrt{\tilde{z}(0)}) - \log(1 - \sqrt{\tilde{z}(0)})\right]$$

This is finite for all finite t.

$\therefore \tilde{z}(t^*)$ can not be equal to 1 in finite t.

⑥

So if $\alpha \tilde{z}(0) < 1$ or $1 < \tilde{z}(0)$

we see $\tilde{z}(t)$ can never get to 1 in finite time and so the tank can never empty ($\tilde{z}(t)$ will always tend to 1 as $t \to \infty$)

4 $$\frac{dT}{dt} = k_b(T_A - T)$$ let t be measured in hours after 10 am

given $T_A = 70$

$T(0) = 80$

$T(1) = 75$

Solve 1st order linear

$$T(t) = c e^{-k_b t} + 70$$

$$80 = c e^{0t} + 70 \Rightarrow c = 10$$

$$T(t) = 10 e^{-k_b t} + 70$$

$$75 = T(1) = 10 e^{-k_b} + 70$$

– over –

⑦

$$\frac{1}{2} = e^{-kb}$$

so we set

$$T(t) = 10\left(e^{-kb}\right)^{t} + 70$$

$$= 10\left(1/2\right)^{t} + 70$$

when is $T(t^*) = 98$?

$$98 = 10\left(1/2\right)^{t^*} + 70$$

$$2.8 = \left(1/2\right)^{t^*}$$

$$\log(2.8) = t^* \log(1/2)$$

$$t^* = \frac{-\log(2.8)}{\log(2)} \approx -1.4854...$$

$\approx$ approx 1 hour & 30 min before 10 am

The guy was killed around

$\boxed{8{:}30 \text{ am}}$

⑧

5) $\frac{dT}{dt} = k_b(T_A - T)$

$\frac{dT}{dt} + k_b T = k_b T_A$

$$\overset{\parallel}{e^{-k_b t}\frac{d}{dt}(e^{k_b t}T)} = k_b T_A$$

So

(*) $\frac{d}{dt}(e^{k_b t}T) = k_b(70 + 10\sin(wt))e^{k_b t}$

where $k_b = \log(1.1)$ $w = \pi/12$

Let's do some integrals

① $\int e^{k_b t}dt = \frac{e^{k_b t}}{k_b}$

② $\int \overset{u}{\sin(wt)}\overset{dv}{e^{k_b t}}dt = I$

by parts twice

$$\overset{\parallel}{\frac{\sin(wt)e^{k_b t}}{k_b}} - \int \frac{w}{k_b}\underset{u}{\cos(wt)}\underset{dv}{e^{k_b t}}dt$$

over

IbyP again

$$= \frac{\sin(wt)\, e^{k_b t}}{k_b} - \frac{w}{k_b}\left(\frac{\cos(wt)\, e^{k_b t}}{k_b} + \int \frac{w \sin(wt)}{k_b} e^{k_b t} \right)$$

$$= \left(\frac{\sin wt}{k_b} - \frac{w}{k_b^2} \cos(wt) \right) e^{k_b t} - \left(\frac{w}{k_b}\right)^2 \underbrace{\int \sin(wt)\, e^{k_b t}\, dt}_{I}$$

So

$$\left(1 + \left(\frac{w}{k_b}\right)^2\right) I = \left(\frac{\sin wt}{k_b} - \frac{w}{k_b^2} \cos wt \right) e^{k_b t}$$

Or

$$I = \frac{1}{1 + \left(\frac{w}{k_b}\right)^2} \left(\frac{k_b \sin wt - w \cos wt}{k_b^2} \right) e^{k_b t}$$

$$= \frac{1}{k_b^2 + w^2} \left(k_b \sin wt - w \cos wt \right) e^{k_b t}$$

Use 1 & 2 in (*) to get

$$e^{k_b t}T(t) - T(0)$$

$$= k_b\left[70\frac{e^{k_b t}}{k_b}\right|_0^t$$

$$+ \frac{10}{k_b^2+w^2}\left(k_b \sin wt - w\cos wt\right)e^{k_b t}\Big|_0^t$$

$$= 70\left(e^{k_b t}-1\right)$$

$$+ \frac{10k_b}{k_b^2+w^2}\left(\left(k_b\sin(wt) - w\cos(wt)\right)e^{k_b t} + w\right)$$

and so we get

$$T(t) = e^{-k_b t}T(0) + 70\left(1-e^{-k_b t}\right)$$

(a) $$+ \frac{10k_b}{k_b^2+w^2}\left(k_b\sin wt - w\cos wt + we^{-k_b t}\right)$$

Wow! I was right.

(11)

(b) For large t $e^{-k_b t} \approx 0$ so

(**) $T(t) \simeq 70 + \dfrac{10 k_b}{k_b^2+w^2}\left(k_b \sin wt - w \cos wt\right)$

$\sin(wt-\varphi)$

$= \sin wt \cos(-\varphi) + \cos wt \sin(-\varphi)$

$= \cos\varphi \sin wt - \sin\varphi \cos(wt)$

(***) But $\dfrac{1}{\sqrt{k_b^2+w^2}}\left(k_b \sin wt - w \cos wt\right)$

So we want $\cos\varphi = \dfrac{k_b}{\sqrt{k_b^2+w^2}}$

$\sin\varphi = \dfrac{w}{\sqrt{k_b^2+w^2}}$.

$\Downarrow$

$\tan\varphi = \dfrac{w/\sqrt{\ }}{k_b/\sqrt{\ }} = \dfrac{w}{k_b} = \dfrac{\pi/12}{\log(1.1)}$

$\simeq 2.74681449\ldots$

over

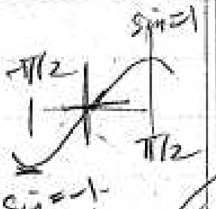
(12)

So $\varphi = \arctan(2.74\ldots)$

$= 1.2216529\ldots$

and so This and (***) into (**) to get

$$T(t) \approx 70 + \frac{10 k_b}{\sqrt{k_b^2 + w^2}} \sin(wt - \varphi)$$



$$\approx 70 + (3.42093\ldots) \sin(wt - \varphi)$$

$\sin(wt - \varphi)$ is max when

$w t_{max} - \varphi = \pi/2$

$$\Rightarrow t_{max} = \frac{\pi/2 + \varphi}{\pi/12} \approx 10.66637\ldots$$

and since $t=0$ corresponds to 10 am

$T_{max} \approx 73.421^\circ$ t_{max} is at around 8:40 pm

$T_{min} = 66.579^\circ$ $w t_{min} - \varphi = -\pi/2$

$$t_{min} = \frac{-\pi/2 - \varphi}{\pi/12} \simeq -1.333629\ldots$$

which is around 840 am.