①

3331 $\boxed{\text{HW 8}}$

1a $\det\begin{pmatrix} 0-\lambda & -1 \\ 2 & 3-\lambda \end{pmatrix} = -\lambda(3-\lambda)+2$

$= \lambda - 3\lambda + 2 = 0$

$\Rightarrow \lambda = \frac{3 \pm \sqrt{9-4\cdot 2}}{2} = \frac{3\pm 1}{2}$

So eigenvalues are $\boxed{\lambda = 1 \; \& \; \lambda = 2}$

b) $\det\begin{pmatrix} 0-\lambda & 2 \\ -1 & 3-\lambda \end{pmatrix} = \lambda - 3\lambda + 2 = 0$

$\boxed{\lambda = 1 \; \& \; \lambda = 2}$

Note, matrix in part b is transpose of matrix in part a,

c) $\det\begin{pmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{pmatrix} = (1-\lambda)(4-\lambda)+2$

$= \lambda^2 - 5\lambda + 6 = 0$

$\lambda = \frac{5 \pm \sqrt{25 - 4\cdot 6}}{2} = \frac{5 \pm 1}{2}$

$\boxed{\lambda = 2, \lambda = 3}$

②

d) $\det\begin{pmatrix} -2-\lambda & -2 \\ 6 & 5-\lambda \end{pmatrix} = (-2-\lambda)(5-\lambda)+12$

$= \lambda^2-3\lambda+2=0$

$\lambda = \frac{3 \pm \sqrt{9-4\cdot 2}}{2}$ $\boxed{\lambda=1,\ \lambda=2}$

2a) $\det\begin{pmatrix} 1-\lambda & 2 \\ 2 & 2-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda)-4$

$= \lambda^2-3\lambda-2=0,\ \lambda = \frac{3 \pm \sqrt{9+8}}{2}$

$\boxed{\lambda = \frac{3-\sqrt{17}}{2},\ \lambda = \frac{3+\sqrt{17}}{2}}$

b) $\det\begin{pmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2+1$

$= \lambda^2-2\lambda+2=0,\ \lambda = \frac{2 \pm \sqrt{4-8}}{2}$

$= \frac{2 \pm 2\sqrt{-1}}{2}$

$\boxed{\lambda = 1-i,\ \lambda = 1+i}$

③

c) $\det\begin{pmatrix} 0-\lambda & -1 & -1 \\ 2 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{pmatrix}$

$= (3-\lambda)\det\begin{pmatrix} -\lambda & -1 \\ 2 & 3-\lambda \end{pmatrix}$

$= (3-\lambda)(\lambda^2 - 3\lambda + 2) = 0$

$\lambda = 3$; $\lambda = \dfrac{3 \pm \sqrt{9-4 \cdot 2}}{2}$

$\boxed{\lambda = 3,\ \lambda = 1,\ \lambda = 2}$

d) $\det\begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & -2 \\ 2 & 2 & 4-\lambda \end{pmatrix}$

$= (1-\lambda)^2(4-\lambda) - \left[2(1-\lambda) - 4(1-\lambda)\right]$

$= (1-\lambda)\left[(1-\lambda)(4-\lambda) + 2\right]$

$= (1-\lambda)\left[\lambda^2 - 5\lambda + 6\right]$ · (error)

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$\lambda = 1$, $\lambda = \frac{5 \pm \sqrt{25-24}}{2}$

So $\boxed{\lambda = 1,\ \lambda = 2,\ \lambda = 3}$

3-1a) $A = \begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix}$ $\lambda = 1, 2$

$\lambda = 1$ $[A - I] = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$r = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda = 2$ $[A - 2I] = \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$

$r = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ $\boxed{R = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}}$

Use cramer's rule to get

$R^{-1} = \frac{1}{-1} \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}}$

$$R^{-1}AR = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \checkmark$$

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3-1c) $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$

$\lambda = 2$; $[A - 2I] = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$

$r = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\lambda = 3$; $[A - 3I] = \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$r = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\boxed{R = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}}$

R^{-1} by Cramer's Rule $\boxed{R^{-1} = \frac{1}{1}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}}$

(over)

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$$R^{-1}AR = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \checkmark$$

3-1d

$$A = \begin{pmatrix} -2 & -2 \\ 6 & 5 \end{pmatrix}$$

$\lambda = 1$: $[A - I] = \begin{bmatrix} -3 & -2 \\ 6 & 4 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 \\ 0 & 0 \end{bmatrix}$

$r = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

$\lambda = 2$: $[A - 2I] = \begin{bmatrix} -4 & -2 \\ 6 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$

$r = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Cramer's Rule

$R = \begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix}$, $R^{-1} = \frac{1}{-1}\begin{pmatrix} -2 & -1 \\ 3 & 2 \end{pmatrix}$

So $R^{-1} = \begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix}$ (oror)

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$$R^{-1}AR = \begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} -2 & -2 \\ 6 & 5 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ -3 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$ ✓

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4-2b) $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$\lambda = 1-i$: $[A-(1-i)I] = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}$

$$\sim \begin{bmatrix} -1 & i \\ -1 & i \end{bmatrix} \sim \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$

$r = \begin{pmatrix} i \\ 1 \end{pmatrix}$ or better for me (mult by $-i$) $r = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

$\lambda = 1+i$ $[A-(1+i)I] = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}$ (over

(8)

$$\sim \begin{bmatrix} 1 & i \\ 1 & i \end{bmatrix} \sim \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$$

$r = \begin{pmatrix} i \\ -1 \end{pmatrix}$ or better for me (mult by $-i$) $r = \begin{pmatrix} 1 \\ i \end{pmatrix}$

So $R = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$, $R^{-1} = \frac{1}{2i}\begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix}$

$$= \frac{1}{2}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

$$R^{-1}AR = \frac{1}{2}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\begin{pmatrix} 1-i & 1+i \\ -1-i & -1+i \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1-i-i+1 & 1+i-i-1 \\ 1-i+i-1 & 1+i+i+1 \end{pmatrix} = \text{(over)}$$

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$$= \begin{pmatrix} 1-i & 0 \\ 0 & 1+i \end{pmatrix}$$

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✓ I may give you a complex problem on your exam, If I do though, it won't be too difficult.

(1-2c) $A = \begin{pmatrix} 0 & -1 & -1 \\ 2 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$\lambda = 1;\ [A-I] = \begin{bmatrix} -1 & -1 & -1 \\ 2 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$

⑨

$$= \begin{pmatrix} 1-i & 0 \\ 0 & 1+i \end{pmatrix}$$

✓ I may give you a complex problem on your exam, If I do though, it won't be too difficult.

4-2c

$$A = \begin{pmatrix} 0 & -1 & -1 \\ 2 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\lambda = 1; \ [A-I] = \begin{bmatrix} -1 & -1 & -1 \\ 2 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$r = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\lambda = 2; \ [A-2I] = \begin{bmatrix} -2 & -1 & -1 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ (over)

(10)

$$\sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad r = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

$$\lambda = 3: \ [A - 3I] = \begin{bmatrix} -3 & -1 & -1 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad r = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

So $R = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -2 & 1 \\ 0 & 0 & -1 \end{pmatrix}$

I usually don't use Cramer's Rule on 3x3's or larger. But here I will so you'll see how it works.

$$R^{-1} = \frac{1}{2-1} \begin{pmatrix} +(2) & -(1) & +(0) \\ -(-1) & +(-1) & -(0) \\ +(1) & -(1) & +(-1) \end{pmatrix}^T$$

cofactor matrix

so $R^{-1} = \begin{pmatrix} 2 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}$

$$R^{-1}AR = \begin{pmatrix} 2 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 & -1 \\ 2 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ -1 & -2 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ -1 & -4 & 3 \\ 0 & 0 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \checkmark$$

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4-2d)

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 2 & 2 & 4 \end{pmatrix}$$

(over)

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$$\lambda=1:\ [A-I]=\begin{bmatrix}0&0&1\\0&0&-2\\2&2&3\end{bmatrix}$$

$$\sim\begin{bmatrix}1&1&0\\0&0&1\\0&0&0\end{bmatrix}\qquad r=\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$\lambda=2:\ [A-2I]=\begin{bmatrix}-1&0&1\\0&-1&-2\\2&2&2\end{bmatrix}$$

$$\sim\begin{bmatrix}1&1&1\\-1&0&1\\0&1&2\end{bmatrix}\sim\begin{bmatrix}1&1&1\\0&1&2\\0&0&0\end{bmatrix}$$

$$z=1\Rightarrow y=-2\Rightarrow x=2-1=1$$

So $r=\begin{pmatrix}1\\-2\\1\end{pmatrix}$

$$\lambda=3:\ [A-3I]=\begin{bmatrix}-2&0&1\\0&-2&-2\\2&2&1\end{bmatrix}$$ (over)

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$$\sim\begin{bmatrix} 2 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$z = 1$, $y = -1$ $\quad 2x = -2(-1) - 1 = 1$

$r = \begin{pmatrix} 1/2 \\ -1 \\ 1 \end{pmatrix}$ Scale (mult by 2) $r = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

So

$$R = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -2 & -2 \\ 0 & 1 & 2 \end{pmatrix}$$

Instead of Cramer's rule, I'll use elimination to get R^{-1}

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & -2 & -2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$ (over)

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$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & -2 & -2 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & -2 & -2 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

so

$$R^{-1} = \begin{pmatrix} 2 & 1 & 0 \\ -2 & -2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$R^{-1}AR =$$

$$\begin{pmatrix} 2 & 1 & 0 \\ -2 & -2 & -1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 2 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ -1 & -2 & -2 \\ 0 & 1 & 2 \end{pmatrix}$$

(over)

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$$=\begin{pmatrix}2 & 1 & 0\\ -2 & -2 & -1\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ -1 & -4 & -6\\ 0 & 2 & 6\end{pmatrix}$$

$$=\begin{pmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{pmatrix} \checkmark$$

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5 in 1-c) $A=\begin{pmatrix}1 & 2\\ -1 & 4\end{pmatrix}$, $\Lambda=\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}$

$R=\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}$ $R^{-1}=\begin{pmatrix}1 & -1\\ -1 & 2\end{pmatrix}$

So $e^{At}=Re^{\Lambda t}R^{-1}$

$$=\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}e^{2t} & 0\\ 0 & e^{3t}\end{pmatrix}\begin{pmatrix}1 & -1\\ -1 & 2\end{pmatrix}$$

$$=\boxed{\begin{pmatrix}2e^{2t}-e^{3t} & -2e^{2t}+2e^{3t}\\ e^{2t}-e^{3t} & -e^{2t}+2e^{3t}\end{pmatrix}}$$

(over)

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Here's a quick check,

Since $e^{A0} = I$ let's see if this agrees with our answer

$$\begin{pmatrix} 2e^0 - e^0 & -2e^0 + 2e^0 \\ e^0 - e^0 & -e^0 + 2e^0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So we're probably right.

5 in 2b)

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad \Lambda = \begin{pmatrix} 1-i & 0 \\ 0 & 1+i \end{pmatrix}$$

$$R = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \quad R^{-1} = \frac{1}{2}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

So

$$e^{At} = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} e^{(1-i)t} & 0 \\ 0 & e^{(1+i)t} \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

$$= \frac{1}{2}e^{t}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{pmatrix}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

(over)

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$$= \frac{e^t}{2}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} e^{-it} & ie^{-it} \\ e^{it} & -ie^{it} \end{pmatrix}$$

$$= \frac{e^t}{2}\begin{pmatrix} e^{-it}+e^{it} & ie^{-it}-ie^{it} \\ -ie^{-it}+ie^{it} & e^{-it}+e^{it} \end{pmatrix}$$

But from Euler's Formulas

$$\frac{e^{it}+e^{-it}}{2} = \cos t \qquad \frac{e^{it}-e^{-it}}{2i} = \sin t$$

use these to get

$$e^{At} = \boxed{e^t\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}}$$

Again, check $e^{A0} = e^0\begin{pmatrix} \cos 0 & \sin 0 \\ -\sin 0 & \cos 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

5 in 2d)

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 2 & 2 & 4 \end{pmatrix} \quad \Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$R = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -2 & -2 \\ 0 & 1 & 2 \end{pmatrix} \quad R^{-1} = \begin{pmatrix} 2 & 1 & 0 \\ -2 & -2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$ (over)

So $e^{At} = R e^{\Lambda t} R^{-1} =$

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & -2 & -2 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} e^{t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ -2 & -2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2e^{t}-2e^{2t}+e^{3t} & e^{t}-2e^{2t}+e^{3t} & -e^{2t}+e^{3t} \\ -2e^{t}+4e^{2t}-2e^{3t} & -e^{t}+4e^{2t}-2e^{3t} & 2e^{2t}-2e^{3t} \\ -2e^{2t}+2e^{3t} & -2e^{2t}+2e^{3t} & -e^{2t}+2e^{3t} \end{pmatrix}$$

Again, it's wise to check

$$e^{A0} = \begin{pmatrix} 2-2+1 & 1-2+1 & -1+1 \\ -2+4-2 & -1+4-2 & 2-2 \\ -2+2 & -2+2 & -1+2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

6a) $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

See matrices in 1c 2 3 5

$A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}, \quad e^{At} = \begin{pmatrix} 2e^{2t}-e^{3t} & -2e^{2t}+2e^{3t} \\ e^{2t}-e^{3t} & -e^{2t}+2e^{3t} \end{pmatrix}$

(over)

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so

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{At}\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} \quad \text{so}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 2e^{2t}-e^{3t} & -2e^{2t}+2e^{3t} \\ e^{2t}-e^{3t} & -e^{2t}+2e^{3t} \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} (2e^{2t}-e^{3t})+2(-2e^{2t}+2e^{3t}) \\ (e^{2t}-e^{3t})+2(-e^{2t}+2e^{3t}) \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} -2e^{2t}+3e^{3t} \\ -e^{2t}+3e^{3t} \end{pmatrix}}$$

6b)

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

See matrices in 2b & 4 & 5

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad e^{At} = e^{t}\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$ (over)

So $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{At} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^t \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= e^t \begin{pmatrix} 2\cos t + \sin t \\ -2\sin t + \cos t \end{pmatrix}$$

$$\boxed{\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^t(2\cos t + \sin t) \\ e^t(-2\sin t + \cos t) \end{pmatrix}}$$

6c)

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 2 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

See matrices in 2d, 4 and 5

$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 2 & 2 & 4 \end{pmatrix} \Rightarrow e^{At} = \begin{pmatrix} 2e^t - 2e^{2t} + e^{3t} \\ \ldots \text{ See page 18} \end{pmatrix}$

②

Compute

$$\begin{pmatrix} 2e^t-2e^{2t}+e^{3t} & e^t-2e^{2t}+e^{3t} & -e^{2t}+e^{3t} \\ -2e^t+4e^{2t}-2e^{3t} & -e^t+4e^{2t}-2e^{3t} & 2e^{2t}-2e^{3t} \\ -2e^{2t}+2e^{3t} & -2e^{2t}+2e^{3t} & -e^{2t}+2e^{3t} \end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix}$$

$$\boxed{\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} 4e^t-9e^{3t}+6e^{3t} \\ -4e^t+18e^{2t}-12e^{3t} \\ -9e^{2t}+12e^{3t} \end{pmatrix}}$$

7 $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

eigenvalues: $\det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2-1=0.$

$\lambda=-1, \quad \lambda=+1$

eigenvectors

$\lambda=-1 \quad [A+I] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad r=\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda=+1 \quad [A-I] = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \quad r=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

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Cramer's Rule

$$R = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad R^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

So

$$e^{At} = R e^{\Lambda t} R^{-1}$$

$$= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^{-t} & -e^{-t} \\ e^{t} & e^{t} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} e^{-t}+e^{t} & -e^{-t}+e^{t} \\ -e^{-t}+e^{t} & e^{-t}+e^{t} \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}}$$

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8a) $\frac{du}{dt} + Au = f(t)$, $u(0) = u_0$

Consider

$$u(t) = e^{-At} u_0 + \int_0^t e^{A(\tau - t)} f(\tau) d\tau$$

From our work

$$\frac{d}{dt} e^{-At} u_0 = -A e^{-At} u_0$$

From Leibniz, also ~~see~~ that

$$\frac{d}{dt} \int_0^t e^{A(\tau - t)} f(\tau) d\tau$$

$$= e^{A(\tau - t)} f(\tau) \Big|_{\tau = t} + \int_0^t \frac{\partial}{\partial t} e^{A(\tau - t)} f(\tau) d\tau$$

$$= \overset{=1}{e^{A0}} f(t) + \int_0^t -A e^{A(\tau - t)} f(\tau)$$

$$= f(t) - A \int_0^t e^{A(\tau - t)} f(\tau) d\tau$$

(over)

(2a)

So $$\frac{du}{dt} = f(t) - A\left(e^{-At}u_0 + \int_0^t e^{-A(\tau-t)} f(\tau)\,d\tau\right)$$

$$= f(t) - A\,u(t)$$

$$\therefore \frac{du}{dt} + Au = f(t)$$

Also
$$u(0) = e^{-A0}u_0 + \int_0^0 e^{A(\tau-t)} f(\tau)\,d\tau$$

$$= Iu_0 + 0 = u_0$$

So $$u(t) = e^{-At}u_0 + \int_0^t e^{A(\tau-t)} f(\tau)\,d\tau$$
solves the IVP.

b) $$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} + \overset{A}{\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ e^t \end{pmatrix}$$

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

over

(25)

$$\boxed{\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}} \qquad \vec{f}(t) = \begin{pmatrix} t \\ e^t \end{pmatrix}$$

From exercise 7

$$A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad e^{-At} = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \vec{0} + \int_0^t \begin{pmatrix} \cosh(t-\tau) & \sinh(t-\tau) \\ \sinh(t-\tau) & \cosh(t-\tau) \end{pmatrix} \begin{pmatrix} \tau \\ e^\tau \end{pmatrix}$$

$$(*) \quad = \int_0^t \begin{pmatrix} \tau\cosh(t-\tau) + e^\tau \sinh(t-\tau) \\ \tau\sinh(t-\tau) + e^\tau \cosh(t-\tau) \end{pmatrix} d\tau$$

Now we've gotta do some integrals.

by Parts

$$\int_0^t \tau\cosh(t-\tau)\,d\tau = \cosh(t) - 1$$

$$\int_0^t \tau\sinh(t-\tau)\,d\tau = -t + \sinh(t)$$

Next use $\sinh(t-\tau) = \frac{1}{2}(e^{(t-\tau)} - e^{-(t-\tau)})$

to get

$$\int_0^t e^\tau \sinh(t-\tau)\,d\tau = \frac{1}{2}te^t - \frac{1}{2}\sinh(t)$$

(over)

(28)

and use $\cosh(t-\tau) = \frac{1}{2}\left(e^{(t-\tau)} + e^{-(t-\tau)}\right)$

to get

$$\int_0^t e^{\tau}\cosh(t-\tau)\,d\tau = \tfrac{1}{2}te^{t} + \tfrac{1}{2}\sinh(t)$$

plug these in to (*) to get

$$\boxed{\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cosh(t) - 1 + \frac{1}{2}te^{t} - \frac{1}{2}\sinh(t) \\ -t + \sinh(t) + \frac{1}{2}te^{t} + \frac{1}{2}\sinh(t) \end{pmatrix}}$$

$\boxed{\text{Check}}$ plug in to check this soln is correct

$$\frac{dx}{dt} - y = \left(\sinh(t) + \cancel{\tfrac{1}{2}te^{t}} + \tfrac{1}{2}e^{t} - \tfrac{1}{2}\cosh(t)\right)$$
$$-\left(-t + \sinh(t) + \cancel{\tfrac{1}{2}te^{t}} + \tfrac{1}{2}\sinh(t)\right)$$

$\boxed{\frac{dx}{dt} - y = t}$?

$$= t + \tfrac{1}{2}e^{t} - \tfrac{1}{2}\left(\frac{e^{t} + \cancel{e^{-t}}}{2}\right) - \tfrac{1}{2}\left(\frac{e^{t} - \cancel{e^{-t}}}{2}\right)$$
$$= t + \tfrac{1}{2}e^{t} - \tfrac{1}{2}e^{t} = t \quad \checkmark$$

? $\frac{dy}{dt} - x = \left(\cancel{-1} + \cancel{\cosh t} + \cancel{\tfrac{1}{2}te^{t}} + \tfrac{1}{2}e^{t} + \tfrac{1}{2}\cosh t\right)$

$\boxed{\frac{dy}{dt} - x = e^{t}}$ $-\left(\cancel{\cosh t} - \cancel{1} + \cancel{\tfrac{1}{2}te^{t}} - \tfrac{1}{2}\sinh t\right)$

(over

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$$= \frac{1}{2}e^t + \frac{1}{2}(\cosh t + \sinh t)$$

$$= \frac{1}{2}e^t + \frac{1}{2}\left(\frac{e^t + \cancel{e^{-t}}}{2} + \frac{e^t - \cancel{e^{-t}}}{2}\right)$$

$$\frac{1}{2}e^t + \frac{1}{2}e^t = e^t \checkmark$$

Finally

$$x(0) = 1 - 1 + \frac{1}{2}0 - \frac{1}{2}0 = 0 \checkmark$$

$$y(0) = -0 + 0 + \frac{1}{2}0 + \frac{1}{2}0 = 0 \checkmark$$