① 3321 — $1^{st}$ order Part I

1a) $\frac{du}{dx} = u^2$ so $\int \frac{du}{u^2} = \int dx$.

$-\frac{1}{u} = x + c \Rightarrow \boxed{u = \frac{-1}{x+c}}$

b) $\frac{du}{dx} = e^{x+u}$ so $\int e^{-u}\,du = \int e^{x}\,dx$

$-e^{-u} = e^{x} + c \Rightarrow \boxed{u = -\log(-c-e^{x})}$

c) $\frac{du}{dx} = u^2+1$ so $\int \frac{du}{u^2+1} = \int dx$

$\arctan(u) = x + c$

$\Rightarrow \boxed{u = \tan(x+c)}$

d) $\frac{du}{dx} = (u^2-u)x$

so $\int \frac{du}{u(u-1)} = \int x\,dx$

Partial fractions $\frac{1}{u(u-1)} = \frac{-1}{u} + \frac{1}{u-1}$

$\int \frac{du}{u(u-1)} = \int \left(\frac{-1}{u} + \frac{1}{u-1}\right) du = \log\left(\left|\frac{u-1}{u}\right|\right) + c$

$\int x\,dx = \frac{x^2}{2} + c$

note here

$\log\left(\left|\frac{u-1}{u}\right|\right) = \frac{x^2}{2} + c \Rightarrow \frac{u-1}{u} = \pm e^{c} e^{x^2/2}$

$= c e^{x^2/2}$

Solve for $u = 1/(1 - c e^{x^2/2})$

②

2a) $\frac{dy}{dx} = \sqrt{|y|}$ , $\int_1^y \frac{dy}{\sqrt{|y|}} = \int_0^x dx$

$y(0)=1$

But as long as $\underline{\underline{y>0}}$ $\int_1^y \frac{dy}{\sqrt{|y|}} = 2\sqrt{y} - 2$

also $\int_0^x dx = x \Rightarrow \boxed{y = \left(\frac{x}{2}+1\right)^2}$

But if $x<-2$ Note that this doesn't solve the ODE! $\left(\frac{x}{2}+1\right) \neq \left|\frac{x}{2}+1\right|$

b) $\frac{dy}{dx} = \sqrt{|y|}$ , $\int_{-1}^y \frac{dy}{\sqrt{|y|}} = \int_0^x$

$y(0)=-1$

But as long as $y<0$ $\int_{-1}^y \frac{dy}{\sqrt{|y|}} = \int_{-1}^y \frac{dy}{\sqrt{-y}} = 2(-\sqrt{-y}+1)$

So $-\sqrt{-y}+1 = \frac{x}{2} \Rightarrow \boxed{y = -\left(1-\frac{x}{2}\right)^2}$

But if $x>2$ This too doesn't solve the ODE b.c. $1-\frac{x}{2} \neq \left|1-\frac{x}{2}\right|$

c) $\frac{dy}{dx} = e^{x-y}$ $\int_0^y e^y dy = \int_0^x e^x dx$

$y(0)=0$

$e^y - 1 = e^x - 1$

$\Rightarrow \boxed{y = x}$

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d) $\frac{du}{dx} = x(u^2+1)$ $\int_0^u \frac{du}{u^2+1} = \int_1^x x\,dx$

$u(1)=0$

$\arctan(u) - 0 = \frac{x^2}{2} - \frac{1}{2}$

So $\boxed{u = \tan\left(\frac{1}{2}(x^2-1)\right)}$

3a) $\frac{du}{dx} + u = x$, $e^{-x}\frac{d}{dx}(e^x u) = x$

So $\int \frac{d}{dx}(e^x u)\,dx = \int x e^x\,dx$ ← I by Parts

$e^x u = c + (x-1)e^x \Rightarrow \boxed{u = ce^{-x} + x - 1}$

b) $\frac{du}{dx} + 2xu = e^{-x^2}$ $\int 2x = x^2$

So $e^{-x^2}\frac{d}{dx}(e^{x^2} u) = e^{-x^2}$

$\int \frac{d}{dx}(e^{x^2} u)\,dx = \int dx$

$e^{x^2} u = c + x$ $\boxed{u = ce^{-x^2} + xe^{-x^2}}$

c) $\frac{du}{dx} - xu = x$ $\int -x\,dx = -x^2/2$

$e^{x^2/2}\frac{d}{dx}(e^{-x^2/2} u) = x$ (over)

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$$\int \frac{d}{dx}\left(e^{-x^2/2}u\right)dx = \underbrace{\int x e^{-x^2/2}\,dx}$$ ← substitution

$$e^{-x^2/2}u = c - e^{-x^2/2} \qquad \boxed{u = ce^{x^2/2} - 1}$$

d) $\frac{du}{dx} + \frac{1}{x}u = x^3 \qquad \int \frac{1}{x} = \log x$

$$e^{-\log x}\frac{d}{dx}\left(e^{\log x}u\right) = x^3 \qquad \text{but } e^{\log x} = x$$

as long as $x > 0$

$$\int \frac{d}{dx}(xu)\,dx = \int x^3 x\,dx$$

$$xu = c + x^5/5 \qquad \boxed{u = \frac{c}{x} + \frac{x^4}{5}}$$

4a) $\frac{du}{dx} - 3u = 0 \qquad \cancel{e^{3x}}\frac{d}{dx}\left(e^{-3x}u\right) = 0$

$u(0) = 1$

$$\int_{x=0}^{x}\frac{d}{dx}\left(e^{-3x}u\right)dx = 0 \qquad e^{-3x}u - e^{0}\cdot 1 = 0$$

$$\boxed{u = e^{3x}}$$

b) $\frac{du}{dx} + 2xu = e^{-x^2} \qquad \cancel{e^{-x^2}}\frac{d}{dx}\left(e^{x^2}u\right) = \cancel{e^{-x^2}}$

$u(0) = 2$

$$\int_0^x \frac{d}{dx}\left(e^{x^2}u\right)dx = \int_0^x 1 \qquad \text{(over)}$$

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$$e^{x^2}u - e^0\cdot 2 = x \qquad \boxed{u = (x+2)e^{-x^2}}$$

c) $\dfrac{du}{dx} - 2u = 1$ $\qquad e^{2x}\dfrac{d}{dx}(e^{-2x}u) = 1$

$u(0) = 3$

$$\int_0^x \frac{d}{dx}(e^{-2x}u)\,dx = \int_0^x e^{-2x}\,dx$$

$$e^{-2x}u - e^0\cdot 3 = \frac{1-e^{-2x}}{2} \qquad \boxed{u = 3e^{2x} + \frac{e^{2x}-1}{2}}$$

d) $\dfrac{du}{dx} + \dfrac{1}{x}u = x^3$ $\qquad \dfrac{1}{x}\dfrac{d}{dx}(xu) = x^3$

$u(1) = 4$

$$\int_1^x \frac{d}{dx}(xu)\,dx = \int_1^x x^4$$

$$xu - 1\cdot 4 = (x^5-1)/5 \qquad \boxed{u = 4/x + \frac{x^5-1}{5x}}$$

5a) $\dfrac{du}{dx} + u = \dfrac{1}{1+e^x}$ $\qquad u(0) = 1$

$$\int_0^x \frac{d}{dx}(e^x u)\,dx = \int_0^x \frac{e^x}{1+e^x}\,dx \qquad \text{Let } 1+e^x = z$$

$$e^x dx = dz$$

$$e^x u - e^0\cdot 1 = \int_2^{1+e^x} \frac{dz}{z} = \log\left(\frac{1+e^x}{2}\right)$$

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$$\boxed{u = e^{-x}\left(1+\log\left(\frac{1+e^x}{2}\right)\right)}$$

b) $\dfrac{du}{dx} = \dfrac{e^{x-u}}{1+e^x}$ $\quad u(1)=0$

$$\int_{u=0}^{u} e^u\,du = \int_1^x \frac{e^x}{1+e^x}\,dx \qquad \begin{aligned} 1+e^x &= z \\ e^x dx &= dz \end{aligned}$$

$$e^u - e^0 = \int_{1+e}^{1+e^x} \frac{dz}{z} = \log\left(\frac{1+e^x}{1+e}\right)$$

$$\boxed{u = \log\left(1+\log\left(\frac{1+e^x}{1+e}\right)\right)}$$

c) $\dfrac{du}{dx} = \dfrac{x}{u}$ $\quad u(0)=1$

$$\int_1^u u\,du = \int_0^x x\,dx \qquad \frac{u^2-1}{2} = \frac{x^2}{2}$$

so $u = + \sqrt{1+x^2}$

Take pos root so that $u(0)=1$

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d) $x\frac{du}{dx} - u = 2x\log x \qquad u(e)=1$

$\frac{du}{dx} - \frac{1}{x}u = 2\log(x)$

$e^{\log x}$ $\frac{d}{dx}\left(e^{-\log x}u\right) = 2\log x \qquad e^{-\log x} = \frac{1}{x}$

$\int_e^x \frac{d}{dx}\left(\frac{1}{x}u\right)dx = \int_e^x \frac{2}{x}\log x\,dx \qquad \log x = z$

$\frac{1}{x}dx = dz$

$\log(e) = 1$

$\frac{1}{x}u - \frac{1}{e}\cdot 1 = 2\int_1^{\log(x)} z\,dz$

$= (\log(x))^2 - 1$

so $\boxed{u = x\left(\frac{1}{e} + (\log(x))^2 - 1\right)}$

e) $\frac{du}{dx} = (x-1)(u^2-1) \qquad u(1)=0$

$\int_0^u \frac{du}{u^2-1} = \int_1^x (x-1)\,dx$

Part Frac $\frac{1}{u^2-1} = \frac{1}{2}\frac{1}{u-1} - \frac{1}{2}\frac{1}{u+1}$

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So $\frac{1}{2}\left[\log\left|\frac{u-1}{-1}\right| - \log\left|\frac{u+1}{1}\right|\right] = \frac{(x-1)^2}{2}$

So $e^{\log\left|\frac{u-1}{u+1}\right|} = e^{(x-1)^2}$

$\left|\frac{u-1}{u+1}\right| = e^{(x-1)^2}$ But $\left|\frac{u-1}{u+1}\right| = \pm\frac{u-1}{u+1}$

But need $-$ since $u(1)=0$

$\frac{1-u}{1+u} = e^{(x-1)^2} \Rightarrow \boxed{u = \frac{1-e^{(x-1)^2}}{1+e^{(x-1)^2}}}$

f) $\frac{du}{dx} + \cos x\, u = \sin x \cos x, \quad u(0)=1$

$\int \cos x\, dx = \sin x$

$e^{-\sin(x)}\frac{d}{dx}\left(e^{\sin(x)}u\right) = \sin x \cos x$

$$\int_{x=0}^{x} \frac{d}{dx}\left(e^{\sin x}u\right) dx = \int_0^x e^{\sin x}\sin x \cos x\, dx$$

$z = \sin x \quad dz = \cos x\, dx$

cover

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1 by Parts
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$$RHS = \int_0^{\sin x} e^{z} z \, dz$$

$$= (z-1)e^{z}\Big|_0^{\sin x} = (\sin x - 1)e^{\sin x} + 1$$

$$LHS = e^{\sin x} u - e^{\sin 0} \cdot 1$$

So $e^{\sin x} u - 1 = (\sin x - 1)e^{\sin x} + 1$

$$\boxed{u = 2e^{-\sin x} + (\sin x - 1)}$$