First order (Part II) — 3321

1a) $\frac{du}{dx} = 1 + \left(\frac{u}{x}\right) + \left(\frac{u}{x}\right)^2$ Let $\frac{u}{x} = v \Rightarrow u = xv$

$\frac{d}{dx}(xv) = 1 + v + v^2$

$x\frac{dv}{dx} + \cancel{v} = 1 + \cancel{v} + v^2, \quad x\frac{dv}{dx} = 1 + v^2$

$\int \frac{dv}{1+v^2} = \int \frac{dx}{x}, \quad \arctan(v) = \log|x| + C,$

$v = \frac{u}{x} = \tan(\log|x| + C)$ | $\boxed{u = x\tan(\log|x| + C)}$

ok here.

b) $\frac{du}{dx} = \frac{x^2 + u^2}{xu} = \frac{1 + \left(\frac{u}{x}\right)^2}{\frac{u}{x}}$ Let $\frac{u}{x} = v$

$\frac{d}{dx}(xv) = \frac{1+v^2}{v}$

This is OK for implicit

$x\frac{dv}{dx} + v = \frac{1+v^2}{v}, \quad x\frac{dv}{dx} = \frac{1+v^2}{v} - \frac{v^2}{v},$

$x\frac{dv}{dx} = \frac{1}{v}, \quad \int v\,dv = \int \frac{dx}{x}$ $\quad \left(\frac{u}{x}\right)^2 = 2\log|x| + c$

$\frac{v^2}{2} = \log|x| + C \quad v^2 = 2\log|x| + 2C$

$v = \pm\sqrt{2\log|x| + c}, \quad \boxed{u = \pm x\sqrt{2\log|x| + c}}$

(3)

c) $\frac{dy}{dx} = \frac{x-y}{x+y} = \frac{1-\frac{y}{x}}{1+\frac{y}{x}}$ let $\frac{y}{x} = v$

$\frac{d}{dx}(xv) = \frac{1-v}{1+v}$, $x\frac{dv}{dx} + v = \frac{1-v}{1+v}$,

$x\frac{dv}{dx} = \frac{1-v}{1+v} - v = \frac{1-v}{1+v} - \frac{v(1+v)}{1+v} = \frac{1-2v-v^2}{1+v}$

$\int \frac{1+v}{1-2v-v^2}\,dv = \int \frac{dx}{x}$

sub. let $w = 1-2v-v^2 \Rightarrow dw = -2(1+v)dv$

$-\frac{1}{2}\int \frac{dw}{w} = \int \frac{dx}{x}$, $-\frac{1}{2}\log|w| = \log|x| + C$

$\log|w| = \log|x|^{-2} + C$ $\quad C = -2C$

$\boxed{\log\left|1-2\left(\frac{y}{x}\right)-\left(\frac{y}{x}\right)^2\right| = \log|x|^{-2} + C}$

Only asked for implicit so this is OK.

But you could have used the quad formula to get the solution in explicit form.

③

d) $\frac{dy}{dx} = \frac{y-x}{x-4y} = \frac{\frac{y}{x}-1}{1-4\frac{y}{x}}$ let $\frac{y}{x} = v$

so $y = xv$

$\frac{d}{dx}(xv) = \frac{v-1}{1-4v}$, $x\frac{dv}{dx} + v = \frac{v-1}{1-4v}$

$x\frac{dv}{dx} = \frac{v-1}{1-4v} - \frac{v(1-4v)}{1-4v} = \frac{4v^2-1}{1-4v}$

$\frac{1-4v}{4v^2-1}\,dv = \frac{dx}{x}$ (*) $\int \frac{1-4v}{(2v-1)(2v+1)}\,dv = \int \frac{dx}{x}$

$= \log|x| + C$

partial fractions

$\frac{1-4v}{(2v-1)(2v+1)} = \frac{A}{2v-1} + \frac{B}{2v+1} = \frac{A(2v+1)+B(2v-1)}{(2v-1)(2v+1)}$

$= \frac{2(A+B)v + (A-B)}{(2v-1)(2v+1)}$

$2(A+B) = -4$

$A-B = 1$

$A = -\frac{1}{2}$

$B = -3/2$

so (*) $\int \frac{1-4v}{(2v-1)(2v+1)}\,dv$

$= \int \frac{-1/2}{2v-1}\,dv + \int \frac{-3/2}{2v+1}\,dv = -\frac{1}{4}\log|2v-1| - \frac{3}{4}\log|2v+1|$

$= -\frac{1}{4}\log\left|(2\frac{y}{x}-1)(2\frac{y}{x}-1)^3\right| = \log|x| + C$

(4)

2a) $\frac{du}{dx} = 1 + \left(\frac{u}{x}\right) + \left(\frac{u}{x}\right)^2$, $u(1) = 1$

Here I want the solution in explicit form.

From 1a $u(x) = x\tan(\log|x| + c)$

so

$1 = 1\tan(\log 1 + c) = \tan(c)$

so $c = \pi/4$

$$\boxed{u(x) = x\tan(\log|x| + \pi/4)}$$

b) $\frac{du}{dx} = \frac{x^2+u^2}{xu}$, $u(1) = 2$

From 1b $u(x) = \pm x\sqrt{2\log|x| + c}$

$2 = u(1) = + \sqrt{2\log 1 + c}$

need + root $2 = \sqrt{c}$

$\Rightarrow c = 4$

so $\boxed{u(x) = + x\sqrt{2\log|x| + 4}}$

c) $\frac{du}{dx} = \frac{x-u}{x+u}$ $u(1) = 0$ From 1c

$\log\left|1 - 2\frac{u}{x} - \left(\frac{u}{x}\right)^2\right| = \log|x|^{-2} + c$

$\log\left|1 - \frac{0}{1} - \left(\frac{0}{1}\right)^2\right| = \log 1 + c$

-over-

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so $c=0$ and get

$$1-2\frac{y}{x}-\left(\frac{y}{x}\right)^2 = x^{-2}$$

use quad form to solve for $\frac{y}{x}$

$$\left(\frac{y}{x}\right)^2+2\left(\frac{y}{x}\right)+x^{-2}-1$$

$$\Rightarrow \left(\frac{y}{x}\right) = \frac{-2\pm\sqrt{4-4(x^{-2}-1)}}{2} = -1\pm\sqrt{2-x^{-2}}$$

But since $y(1)=0$ need $+$ root so

$$\boxed{y = x\left(-1+\sqrt{2-x^{-2}}\right)}$$

2a) $\frac{dy}{dx} = \frac{x-y}{x}$, $y(1)=0$ This is a new ODE

$$\frac{d}{dx}(xv) = 1-v, \quad x\frac{dv}{dx}+v = 1-v$$

So $x\frac{dv}{dx} = 1-2v$ I'm gonna use def integrals to do this one

$v(1) = 1\cdot y(1) = 0$

$$\int_0^{y/x}\frac{dv}{1-2v} = \int_1^x\frac{dx}{x}$$

$$-\frac{1}{2}\log(|1-2v|)+0 = \log(x)-0$$ (over)

⑥

$$\log|1-2v| = \log x^{-2}$$

$$\left|1-2\frac{u}{x}\right| = x^{-2}$$

$$2\frac{u}{x}-1 = \pm x^{-2}$$ But since $u(1)=0$ need $-$ root

$$2\frac{u}{x} = 1-x^{-2}$$

$$\boxed{u = \frac{1}{2}\left(x-x^{-1}\right)}$$

3a) $$(u+x)\frac{du}{dx} + u+2x^3 = 0$$

$$\frac{\partial}{\partial x}(u+x) \stackrel{?}{=} \frac{\partial}{\partial u}(u+2x^3)$$

$$1 = 1 \checkmark$$ exact,

$$\frac{\partial \Lambda}{\partial u} = u+x \Rightarrow \Lambda = \int(u+x)\,du + h(x)$$

$$\Lambda = u^2/2 + xu + h(x)$$

$$\frac{\partial \Lambda}{\partial x} = u + h'(x) = u+2x^3 \Rightarrow h(x) = \frac{x^4}{2}$$

so $$\Lambda = \frac{u^2}{2} + xu + \frac{x^4}{2} = \text{const}$$

implicit relation (over)

④ Try to solve for u explicitly

$$u^2 + 2xu + x^4 - c = 0$$

Use quad formula

$$u = \frac{-2x \pm \sqrt{(2x)^2 - 4(x^4 - c)}}{2}$$

$$\boxed{u = -x \pm \sqrt{x^2 - x^4 + c}}$$

b) $x e^u \frac{du}{dx} + e^u + 1 = 0$

$$\frac{\partial}{\partial x}(x e^u) \stackrel{?}{=} \frac{\partial}{\partial u}(e^u + 1)$$

$$e^u = e^u \checkmark \text{ exact.}$$

$$\Lambda_u = x e^u \Rightarrow \Lambda = \int x e^u \, du + h(x)$$

$$= x e^u + h(x)$$

$$\frac{\partial}{\partial x}(x e^u + h(x)) = e^u + h'(x) = e^u + 1$$

$$u = \log\left(\frac{c - x}{x}\right)$$

$$\therefore h'(x) = 1 \Rightarrow h(x) = x$$

$$\Lambda = x e^u + x = \text{const} \Rightarrow e^u = \frac{c - x}{x}$$

(8)

c) $(2xu+2)\frac{du}{dx} + u^2 + 4x^3 = 0$

$$\frac{\partial}{\partial x}(2xu+2) \overset{?}{=} \frac{\partial}{\partial u}(u^2+4x^3)$$

$$\begin{matrix} \| & & \| \\ 2u & = & 2u \end{matrix} \checkmark \text{ exact.}$$

$$\frac{\partial \Lambda}{\partial u} = 2xu+2 \Rightarrow \Lambda = \int(2xu+2)\,du + h(x)$$

$$= xu^2 + 2u + h(x)$$

$$\frac{\partial \Lambda}{\partial x} = \frac{\partial}{\partial x}(xu^2+2u+h(x)) = u^2+4x^3$$

$$u^2 + h'(x) \Rightarrow h'(x) = 4x^3 \text{ so } h(x) = x^4$$

$$\therefore \Lambda = xu^2 + 2u + x^4 = \text{const}$$

Again use quad formula to get exact representation

$$u = \frac{-2 \pm \sqrt{4 - 4x(x^4 - c)}}{2x}$$

$$= -\frac{1}{x} \pm \frac{\sqrt{1 - x(x^4 - c)}}{x}$$

or $\boxed{u(x) = -\frac{1}{x} \pm \sqrt{\frac{1}{x^2} - \frac{1}{x}(x^4 - c)}}$

①

d) $(u^3 - x^2 u)\frac{du}{dx} - xu^2 = 0$

$$\frac{\partial}{\partial x}(u^3 - x^2 u) \overset{?}{=} \frac{\partial}{\partial u}(-xu^2)$$

$$\begin{array}{cc} \parallel & \parallel \\ -2xu & -x2u \end{array} \checkmark \text{ exact.}$$

$$\frac{\partial \Lambda}{\partial u} = u^3 - x^2 u \Rightarrow \Lambda = \int (u^3 - x^2 u)\,du + h(x)$$

$$= \frac{u^4}{4} - x^2\frac{u^2}{2} + h(x)$$

So need

$$\frac{\partial}{\partial x}\left(\frac{u^4}{4} - x^2\frac{u^2}{2} + h(x)\right) = -xu^2 + h'(x) = -xu^2$$

$$h'(x) = 0$$

So

$$\frac{u^4}{4} - \frac{1}{2}x^2u^2 = \text{const.}$$ Implicit

$$u^4 - 2x^2u^2 - c = 0$$ use quad formula to get u^2

$$u^2 = \frac{2x^2 \pm \sqrt{4x^4 + 4c}}{2} = x^2 \pm \sqrt{x^4 + c}$$

$$\Rightarrow \boxed{u = \pm\sqrt{x^2 \pm \sqrt{x^4 + c}}}$$

(10)

4a) $x\frac{dy}{dx} + (y+x) = 0$

Exact? $\frac{\partial x}{\partial x} \stackrel{?}{=} \frac{\partial}{\partial y}(y+x)$, $1=1$ yes — exact.

Seperable? $\frac{dy}{dx} = -\left(\frac{y+x}{x}\right)$ No — not sep

Homogeneous? $\frac{dy}{dx} = -\left(\frac{y}{x}+1\right)$ yes — homogen.

Linear? $\frac{dy}{dx} + \frac{1}{x}y = -1$ yes linear

b) $y\frac{dy}{dx} + y+x = 0$

$\frac{\partial}{\partial x}y \stackrel{?}{=} \frac{\partial}{\partial y}(y+x)$ $0 \neq 1$ Not exact

$\frac{dy}{dx} = -\left(\frac{y+x}{y}\right)$ nope. Not seperable

$\frac{dy}{dx} = -\left(\frac{y/x+1}{y/x}\right)$ is homogen

$\frac{dy}{dx} + \frac{x}{y} + 1 = 0$ Not linear

NO

(11)

c) $(x^2+1)\frac{dy}{dx} + y^2 = 0$

$\frac{\partial}{\partial x}(x^2+1) \stackrel{?}{=} \frac{\partial}{\partial y} y^2 \qquad 2x \neq 2y$ NOT EXACT

$(x^2+1)\frac{dy}{dx} = -y^2$

$\frac{dy}{y^2} = -\frac{dx}{x^2+1}$ is separable

$\frac{dy}{dx} = -\frac{y^2}{x^2+1}$ NOT homogeneous

$\frac{dy}{dx} + \frac{1}{x^2+1} y^2 = 0$ Nope NOT Linear

d) $(y^2+1)\frac{dy}{dx} + e^{x+y} = 0$

$\frac{\partial}{\partial x}(y^2+1) \stackrel{?}{=} \frac{\partial}{\partial y} e^{x+y}$ Nope NOT exact

$(y^2+1)\frac{dy}{dx} = -e^x e^y$

$(y^2+1) e^{-y}\, dy = -e^x dx$ is separable

you must say more than this on exam

clearly NOT homogeneous

clearly NOT Linear

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e) $(y^2+x^2)\frac{dy}{dx} + e^y = 0$

$\frac{\partial}{\partial x}(y^2+x^2) \overset{?}{=} \frac{\partial}{\partial y} e^y$ nope NOT Exact

$\frac{dy}{dx} = -\frac{e^y}{y^2+x^2}$ NOT seperable

$\frac{dy}{dx} + \frac{e^y}{y^2+x^2} = 0 \longleftarrow$ nope NOT homogeneous

NOT Linear

f) $(y+x+2)\frac{dy}{dx} + (y+x+1) = 0$

$\frac{\partial}{\partial x}(y+x+2) \overset{?}{=} \frac{\partial}{\partial y}(y+x+1)$ $1 = 1$ ✓ is Exact

$\frac{dy}{dx} = -\frac{y+x+1}{y+x+2}$ NOT seperable

$\frac{dy}{dx} = -\frac{y/x+1+\frac{1}{x}}{y/x+1+\frac{2}{x}}$ NOT homogeneous

clearly NOT Linear

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g) $(x^2+1)\frac{du}{dx} + xu + x^2 = 0$

$\frac{\partial}{\partial x}(x^2+1) \stackrel{?}{=} \frac{\partial}{\partial u}(xu+x^2)$ NOT exact

$2x \neq x$ nope

$\frac{du}{dx} = -\frac{x(u+x)}{x^2+1}$ NOT separable

$\frac{du}{dx} = \frac{-x^2(u/x+1)}{x^2+1}$ NOT homogeneous

$\frac{du}{dx} + \frac{x}{x^2+1}u = -\frac{x^2}{x^2+1}$ is linear

h) $(u+x)\frac{du}{dx} + 2u + 2 = 0$

$\frac{\partial}{\partial x}(u+x) \stackrel{?}{=} \frac{\partial}{\partial u}(2u+2)$ $1 \neq 2$ NOT exact

$\frac{du}{dx} = -\frac{2(u+1)}{u+x}$ NOT separable

you say more. NOT homogeneous

clearly NOT linear

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5a) $\frac{dy}{dx} + u = u^4$

Let $u = v^{\gamma}$

$\left(\gamma v^{\gamma-1}\frac{dv}{dx} + v^{\gamma} = v^{4\gamma}\right) \Big/ v^{\gamma-1}$

$\gamma \frac{dv}{dx} + v = v^{4\gamma-\gamma+1}$ let $3\gamma = -1$ or $\gamma = -1/3$ to set

$-\frac{1}{3}\frac{dv}{dx} + v = 1$

or $\frac{dv}{dx} - 3v = -3$

$\int \frac{d}{dx}(e^{-3x}v) = \int 3e^{-3x} = e^{-3x} + c$

$\parallel$

$e^{-3x}v \Rightarrow v = ce^{3x} + 1$

But $u = v^{-1/3} \Rightarrow \boxed{u = \frac{1}{\sqrt[3]{ce^{3x}+1}}}$

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b) $$x^2\frac{du}{dx} + xu = -u^2$$

$$\frac{du}{dx} + \frac{1}{x}u = -\frac{1}{x^2}u^2$$

Let $u = v^\gamma$

$$\gamma v^{\gamma-1}\frac{dv}{dx} + \frac{1}{x}v^\gamma = -\frac{1}{x^2}v^{2\gamma}$$

$$\gamma\frac{dv}{dx} + \frac{1}{x}v = -\frac{1}{x^2}v^{2\gamma-\gamma+1} \qquad \boxed{\text{Let } \gamma = -1}$$

$$\frac{dv}{dx} - \frac{1}{x}v = +\frac{1}{x^2}$$

$$\Downarrow$$

$$x\frac{d}{dx}\left(\frac{1}{x}v\right) = \frac{1}{x^2} \qquad \int \frac{d}{dx}\left(\frac{1}{x}v\right) = \int x^{-3}$$

$$\frac{1}{x}v = c - \frac{x^{-2}}{2}$$

$$v = xc - \frac{x^{-1}}{2}$$

$$\boxed{u = v^{-1} = \frac{1}{cx - \frac{1}{2}x^{-1}} = \frac{x}{cx^2 - 1/2}}$$

(b)

6 $(u+x)\frac{du}{dx} + 2(u+1) = 0$

a) Let $u+1 = v$ and $x-1 = y$

$u = v-1$ $\quad$ $x = y+1$

$$\frac{d}{dx} = \frac{dy}{dx}\frac{d}{dy} = \frac{d(x-1)}{dx}\frac{d}{dy} = \frac{d}{dy}$$

$$\frac{du}{dx} = \frac{d(v-1)}{dy} = \frac{dv}{dy}$$

Plug in

$$((v-1)+(y+1))\frac{dv}{dy} + 2(v-1+1) = 0$$

so

$$(v+y)\frac{dv}{dy} + 2v = 0$$

This is homogenous in v & y

b) $$\frac{dv}{dy} = -\frac{2v}{v+y} = -2\frac{v/y}{v/y+1}$$

Let $w = \frac{v}{y}$, $v = yw$

$$\frac{d}{dy}(yw) = -2\frac{w}{w+1}$$

(over)

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$$y \frac{dW}{dy} + W = -2 \frac{W}{W+1}$$

$$y \frac{dW}{dy} = -\frac{2W}{W+1} - \frac{W(W+1)}{W+1} = -\frac{W^2+3W}{W+1}$$

So

$$* - \int \frac{W+1}{W^2+3W} \, dW = \int -\frac{dy}{y}$$

use partial fraction

$$\frac{W+1}{W(W+3)} = \frac{A}{W} + \frac{B}{W+3} = \frac{A(W+3) + BW}{W(W+3)}$$

$$= (A+B)W + 3A$$

$A+B=1$

$3A = 1 \Rightarrow A = 1/3 \quad B = 2/3$

$$* = \frac{1}{3} \log|W| + \frac{2}{3} \log(W+3) = -\log|y| + C$$

$$\log|W(W+3)^2| = \log|y|^{-3} + C$$

(over)

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$$|w(w+3)^2| = e^{c'}/|y|^3 \qquad \text{Let } \pm e^c = \tilde{c}$$

$$\Rightarrow w(w+3)^2 = \frac{\pm e^c}{|y^3|} = \frac{\tilde{c}}{y^3}$$

$$w = \frac{v}{y}$$

$$\frac{v}{y}\left(\frac{v}{y}+3\right)^2 = \frac{c}{y^3}$$

$$v(v+3y)^2 = c$$

Finally $v = u+1$ and $y = x-1$ and get

$$(u+1)(u+1+3(x-1))^2 = c$$

$$\phi: \boxed{(u+1)(u-2+3x)^2 = c}$$