①

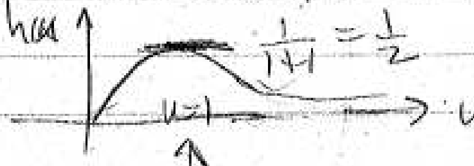
Gronwall and 1st order systems

1) $\dfrac{du}{dt} = \dfrac{u \sin(u)}{u^2+1}$, $u(0)=1$

$$\left| \frac{u \sin(u)}{u^2+1} \right| \leq \frac{|u|}{u^2+1}$$

But $h(u) = \dfrac{u}{u^2+1}$, $h'(u) = \dfrac{(u^2+1) - u(2u)}{u^2+1}$

$= 0$ when $\underline{u = \pm 1}$



So $\left| \dfrac{u \sin(u)}{u^2+1} \right| \leq \dfrac{1}{2}$

So growth condition is satisfied

with $a(T) = 0 \quad b(T) = 1/2$

$\Rightarrow |u(t)| \leq \left(\overset{|u_0|}{1} + \frac{1}{2}T \right) e^{0|t-0|}$

$$= \left(1 + \frac{1}{2}T\right)$$

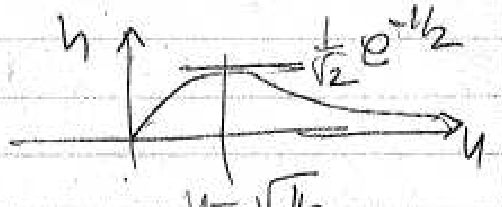
for all $|t| \leq T$

(2)

2) $\frac{du}{dt} = (tu+1)e^{-u^2}$, $u(0)=2$

$$\left|(tu+1)e^{-u^2}\right| \leq t|u|e^{-u^2} + 1$$

put $h(u) = ue^{-u^2}$



$h'(u) = (-2u^2+1)e^{-u^2} = 0$

when $u^2 = 1/2$

So $$\left|(tu+1)e^{-u^2}\right| \leq t\frac{1}{\sqrt{2}}e^{-1/2} + 1$$

Use growth condition with

$$a(T) = 0 \qquad b(T) = \frac{T}{\sqrt{2}}e^{-1/2} + 1$$

$$\Longrightarrow |u(t)| \leq \Big(\underbrace{2}_{|u_0|} + \Big(\frac{Te^{-1/2}}{\sqrt{2}} + 1\Big)T\Big)e^{0|t|}$$

$$= 2 + \Big(\frac{Te^{-1/2}}{\sqrt{2}} + 1\Big)T$$

for all $|t| \leq T$,

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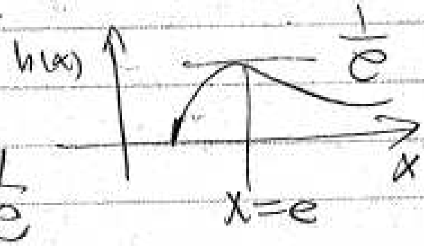
3) $\frac{du}{dt} = \log(t^2u^2+1)$, $u(0)=3$

$$\left|\log(t^2u^2+1)\right| = \frac{2\log\left(\sqrt{t^2u^2+1}\right)}{\sqrt{t^2u^2+1}}\sqrt{t^2u^2+1}$$

Notice that $h(x) = \frac{\log x}{x}$
for $x \geq 1$

$$h'(x) = \frac{x\frac{1}{x} - \log(x)}{x^2} = 0$$

$$\Rightarrow x = e$$



$$\Rightarrow h(x) \leq \frac{\log(e)}{e} = \frac{1}{e}$$
$\forall x \geq 1$

Also $\sqrt{t^2u^2+1} \leq \sqrt{t^2u^2+2|tu|+1}$

$$= \sqrt{(|tu|+1)^2}$$

$$= |tu|+1$$

whoops

So $\left|\log(t^2u^2+1)\right| \leq \frac{2}{e}(|tu|+1)$

④

So use $a(T) = \frac{2}{e}T$ $b(T) = \frac{2}{e}$

in the growth condition to get

$$|u(t)| \leq \left(3 + \frac{2}{e}T\right) e^{\left(\frac{2}{e}T\right)|t|}$$

for all $|t| \leq T$.

4) $\frac{du}{dt} = \frac{u^3 + t^3}{u^2+1}$, $u(0) = 4$

$$\left|\frac{u^3+t^3}{u^2+1}\right| \leq \frac{|u|^3}{u^2+1} + \frac{|t|^3}{u^2+1}$$

$$= \left(\frac{u^2}{u^2+1}\right)|u| + \frac{|t|^3}{u^2+1}$$

$$\leq |u| + |t|^3$$

use in growth cond

$a(T) = 1$ $b(T) = T^3$

So $|u(t)| \leq \left(\underset{|u_0|}{4} + T^3 \cdot T\right) e^{|t|} = (4+T^4)e^{|t|}$

for all $|t| \leq T$

5) $$\frac{d^2u}{dt^2} + \frac{2}{t}\frac{du}{dt} + \frac{1}{t^2}u = e^t$$

$u(1)=1 \quad u_t(1)=2$

Let $\frac{du}{dt} = v \Rightarrow \frac{d^2u}{dt^2} = \frac{dv}{dt}$

So $$\frac{du}{dt} = v$$

$$\frac{dv}{dt} + \frac{2}{t}v + \frac{1}{t^2}u = e^t$$

or
$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -\frac{2}{t}v - \frac{1}{t^2}u + e^t \end{pmatrix}$$

$$\boxed{= \vec{f}(u,v,t)}$$

with initial conds

$$\begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

⑥

b) $\frac{d^2u}{dt^2} - t^2\frac{du}{dt} + t^4u + \cos(t) = 0$

$u(0) = 2 \quad u_t(0) = 3$

Let $\frac{du}{dt} = v \Rightarrow \frac{d^2u}{dt^2} = \frac{dv}{dt}$

$$\frac{du}{dt} = v$$

$$\frac{dv}{dt} - t^2v + t^4u + \cos t = 0$$

or $\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ t^2v - t^4u - \cos t \end{pmatrix}$

with initial conditions

$$\begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

⑦

7) $\frac{d^3y}{dt^3} - 2\frac{d^2y}{dt^2} - 3\frac{dy}{dt} - 4y = 0$

$y(0) = 1 \quad y_t(0) = 2, \quad y_{tt}(0) = 3$

Let $\frac{dy}{dt} = v \quad \frac{dv}{dt} = w \quad \frac{dw}{dt} = \frac{d^3y}{dt^3}$

$= \frac{d^2y}{dt^2}$

So $\frac{dy}{dt} = v$

$\frac{dv}{dt} = w$

$\frac{dw}{dt} - 2w - 3v - 4y = 0$

or $\frac{d}{dt}\begin{pmatrix} y \\ v \\ w \end{pmatrix} = \begin{pmatrix} v \\ w \\ 2w + 3v + 4y \end{pmatrix}$

with initial conditions

$\begin{pmatrix} y(0) \\ v(0) \\ w(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

⑧

8) $$\frac{d^4u}{dt^4} = u\frac{d^2u}{dt^2}$$ $u(5)=1$

$u_t(5)=u_{tt}(5)$

$= u_{ttt}(5) = 0$

Let $\frac{du}{dt} = v$ $\frac{dv}{dt} = w$ $\frac{dw}{dt} = x$

$\frac{d^2u}{dt^2}$ $\frac{d^2v}{dt^2} = \frac{d^3u}{dt^3}$

So $\frac{du}{dt} = v$

$\frac{dv}{dt} = w$

$\frac{dw}{dt} = x$

$\frac{dx}{dt} = uw$

$$\rightarrow \frac{d}{dt}\begin{pmatrix} u \\ v \\ w \\ x \end{pmatrix} = \begin{pmatrix} v \\ w \\ x \\ uw \end{pmatrix}$$

$$\begin{pmatrix} u(5) \\ v(5) \\ w(5) \\ x(5) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

(9)

a) write as a first order system

$$\begin{cases} \dfrac{d^2u}{dt^2} + a(t)\dfrac{du}{dt} + b(t)u = f(t) \\ u(t_0) = u_0 \; ; \quad u_t(t_0) = u_1 \end{cases}$$

$$\frac{du}{dt} = v \Rightarrow \frac{d^2u}{dt^2} = \frac{dv}{dt}$$

$$\frac{d}{dt}u = v$$

$$\frac{dv}{dt} = -a(t)\,v - b(t)u + f(t)$$

or

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -a(t)v - b(t)u + f(t) \end{pmatrix}$$

with $\begin{pmatrix} u(t_0) \\ v(t_0) \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$ $\qquad \dfrac{d\vec{v}}{dt} = \vec{f}(\vec{u}, t)$

(over)

(10)

$$\|f\| = \left\| \begin{pmatrix} v \\ -a(t)v - b(t)u + f(t) \end{pmatrix} \right\|$$

using the one-norm

$$= |v| + |a(t)v + b(t)u - f(t)|$$

$$\leq |v| + |a(t)||v| + |b(t)||u| + |f(t)|$$

$$\leq |b(t)||u| + (1+|a(t)|)|v| + |f(t)|$$

$$\leq \max\left(|b(t)|, (1+|a(t)|)\right)\left(|u| + |v|\right) + |f(t)|$$

$$= \max(|b(t)|, (1+|a(t)|)) \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\| + |f(t)|$$

So $\alpha(T) = \max\limits_{|t| \leq T} \left(\max |b(t)|, (1+|a(t)|) \right)$

in growth condition

(One)

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and $b(t) = \max_{|t| \le T} |f(t)|$

in growth condition

$$\left\| \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} \right\| = |u_0| + |u_1|$$

one norm

So

$$|u| + |u_t| = \left\| \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \right\|$$

$$\le \left(|u_0| + |u_1| + \max_{|t| \le T} |f(t)| \right)$$

$$\exp\left(\max_{|t| \le T} \left(|b(t)|, (1 + |a(t)|) \right) |t - 0| \right)$$

and since $a(t)$, $b(t)$ & $f(t)$ are continuous these are all bounded for bounded T.

(12)

10) $\frac{du}{dt} = \sqrt[3]{u}$

(a) $u(0) = 0$, Clearly $u(t) = 0$ is a solution to this IVP.

But also, separate

$$\int_0^u \frac{du}{u^{1/3}} = \int_0^t dt = t$$

$$\| $$

$$\frac{u^{2/3}}{2/3} - 0$$

So $u^{2/3} = \frac{2}{3} t$

Always pos

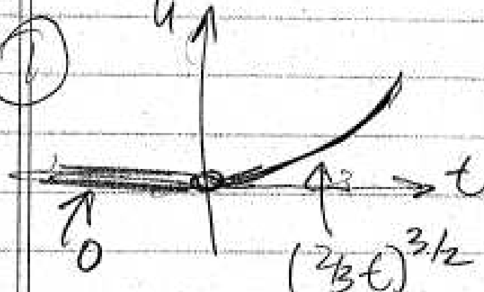
$\therefore$ $u = \left(\frac{2}{3} t\right)^{3/2} \geq 0$

But is valid only for $t \geq 0$.

⑬

So we get two continuously differentiable solutions

1



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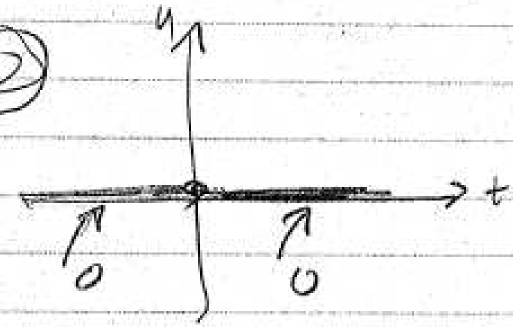

It's easy to see $f(u) = \sqrt[3]{u}$ is NOT Lipschitz around $u=0$. Suppose it were, then

$$L \geq \frac{|f(u)-f(v)|}{|u-v|} \quad \forall u,v \text{ in a nbhd of } u=0$$

$$= \frac{|\sqrt[3]{u} - \sqrt[3]{v}|}{|u-v|} = \frac{|\varepsilon^{1/3}|}{|\varepsilon|} = \frac{1}{|\varepsilon|^{2/3}}$$

Let $v=0$ and $u=\varepsilon$

But since $\frac{1}{|\varepsilon|^{2/3}}$ blows up as $\varepsilon \to 0$ we see this is impossible.

now solve

b) $\frac{du}{dt} = \sqrt[3]{u}$

$u(0) = 1$

$$\int_1^u \frac{du}{\sqrt[3]{u}} = \int_0^t dt = t$$

$$\frac{u^{2/3} - 1^{2/3}}{2/3}$$

This is the only one and one solution

so $u^{2/3} = 1 + \frac{2}{3}t$

Only valid as long as $1 + 2/3\, t \geq 0$

$\Rightarrow$ $u(t) = \left(1 + 2/3\, t\right)^{3/2}$

which is valid provided $\Rightarrow$

$$-3/2 < t < \infty$$

This contains $t=0$.

This time however $\sqrt[3]{u}$ **is** Lipshitz near $u = 1$. (it's diffbl there)

(15)

ii) $\frac{du}{dt} = \sqrt{|1-u^2|}$

a) $u(0) = 1$

Case when $|u| \leq 1$ seperate

$$\int_1^u \frac{du}{\sqrt{1-u^2}} = \int_0^t dt = t$$

But $\frac{d}{du} \arcsin(u) = \frac{1}{\sqrt{1-u^2}}$ so

$$\int_1^u \frac{du}{\sqrt{1-u^2}} = \arcsin(u) - \arcsin(1) = t$$

$$= \arcsin(u) - \pi/2$$

So $\arcsin(u) = \pi/2 + t$

$\Rightarrow \boxed{u(t) = \sin(\pi/2 + t) = \cos(t)}$

Case when $|u| \geq 1$ seperate

$$\int_1^u \frac{du}{\sqrt{u^2-1}} = \int_0^t = t$$

(over)

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But $\frac{d}{du}\operatorname{arccosh}(u) = \frac{1}{\sqrt{u^2-1}}$ so

$$\int_1^u \frac{du}{\sqrt{u^2-1}} = \int_0^t dt = t$$

$$\operatorname{arccosh}(u) - \operatorname{arccosh}(1) = t$$

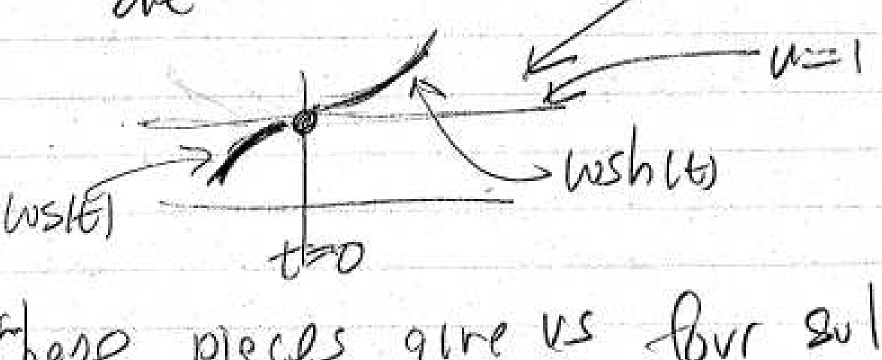
$$= \operatorname{arccosh}(u) - 0 = t$$

so $\boxed{u(t) = \cosh(t)}$

we also have the obvious solution

$\boxed{u(t) = 1}$

Now $\frac{du}{dt} = \sqrt{|1-u^2|} \geq 0$

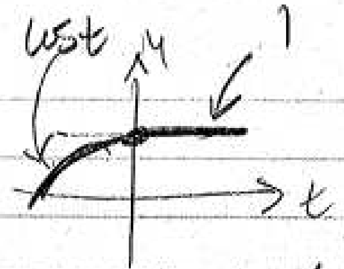


$\frac{du}{dt} = \sin$

these pieces give us four solutions (ones)

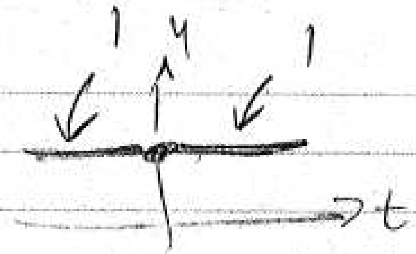
(17)

(1)



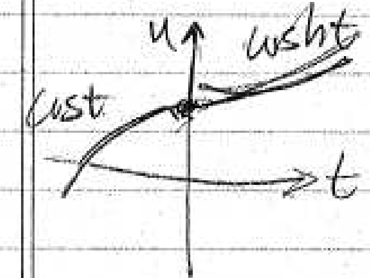
$$u(t)=\begin{cases}\cos t & t\le 0\\ 1 & t\ge 0\end{cases}$$

(2)



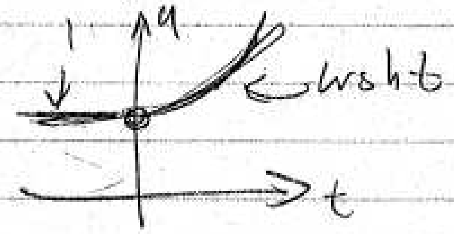
$$u(t)=\begin{cases}1 & t\le 0\\ 1 & t\ge 0\end{cases}$$

(3)



$$u(t)=\begin{cases}\cos t & t\le 0\\ \cosh t & t\ge 0\end{cases}$$

(4)



$$u(t)=\begin{cases}1 & t\le 0\\ \cosh t & t\ge 0\end{cases}$$

These are all continuously differentiable solutions in a neighborhood of $u(0)=1$

$f(u)=\sqrt{|1-u^2|}$ is NOT Lipschitz around $u=1$. Suppose it were

$$L \ge \left|\frac{f(u)-f(v)}{u-v}\right| = \left|\frac{\sqrt{|1-u^2|}-\sqrt{|1-v^2|}}{u-v}\right|$$

(over)

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Take $v=1$ and $u=1+\varepsilon$

$$L \geq \left| \frac{\sqrt{|\varepsilon^2+2\varepsilon|} - 0}{1+\varepsilon - 1} \right|$$

$$= \frac{\sqrt{|\varepsilon|}}{|\varepsilon|}\sqrt{2+\varepsilon} = \frac{\sqrt{2+\varepsilon}}{\sqrt{|\varepsilon|}}$$

But this blows up as $\varepsilon \to 0$.

So $L \geq \left|\frac{f(u)-f(v)}{u-v}\right|$ for all

u, v near $u=1$ is impossible.

b) $\frac{du}{dt} = \sqrt{1-u^2}$

$u(0)=0$

Only need consider case $|u|<1$

$$\int_0^u \frac{du}{\sqrt{1-u^2}} = \int_0^t dt = dt$$

$$\arcsin(u) - \underbrace{\arcsin(0)}_{0} = t$$ (over)

So $\boxed{u(t) = \sin t}$

Clearly $f(u) = \sqrt{1-u^2}$ is differentiable around $u=0$, $\therefore$ $f(u)$ is Lipschitz and we get, as per Theorem, one solution only.