① Intro Solutions

1a $D(y) \equiv y\frac{dy}{dx}$

Try $D(cy) = (cy)\frac{d}{dx}cy = c^2 y\frac{dy}{dx}$

$\neq c\,D(y)$

So This is nonlinear, It's <u>first</u> order

b) $D(y) \equiv y + \frac{dy}{dx}$

$D(y_1+y_2) = (y_1+y_2) + \frac{d}{dx}(y_1+y_2)$

$= y_1+y_2 + \frac{dy_1}{dx} + \frac{dy_2}{dx}$ ✓

$= y_1 + \frac{dy_1}{dx} + y_2 + \frac{dy_2}{dx} = D(y_1) + D(y_2)$

$D(cy) = cy + \frac{d}{dx}cy = c\left(y + \frac{dy}{dx}\right)$

$= c\,D(y)$ ✓

So This is Linear.

It's also <u>first</u> order

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c) $D(y) \equiv x^2 \frac{d^2y}{dx^2} + \frac{dy}{dx}$

$D(y_1+y_2) = x^2 \frac{d^2}{dx^2}(y_1+y_2) + \frac{d}{dx}(y_1+y_2)$

$= x^2\left(\frac{d^2y_1}{dx^2} + \frac{d^2y_2}{dx^2}\right) + \frac{dy_1}{dx} + \frac{dy_2}{dx}$

$= x^2\frac{d^2y_1}{dx^2} + \frac{dy_1}{dx} + x^2\frac{d^2y_2}{dx^2} + \frac{dy_2}{dx}$

$= D(y_1) + D(y_2)$ ✓

$D(cy) = x^2\frac{d^2}{dx^2}(cy) + \frac{d}{dx}(cy)$

$= c\left(x^2\frac{d^2y}{dx^2} + \frac{dy}{dx}\right) = c\,D(y)$ ✓

So This is lineer, It's also <u>second</u> order

d) $D(y) \equiv \frac{dy}{dx} + y^2$

$D(cy) = \frac{d}{dx}(cy) + (cy)^2$

$= c\frac{dy}{dx} + c^2y^2 \neq c\,D(y)$

②

So it's nonlinear.

It's first order

a) $\frac{d^2y}{dx^2} + y\frac{dy}{dx} + y = 0$

Let $D(y) = \frac{d^2y}{dx^2} + y\frac{dy}{dx} + y$

But $D(cy) = c\frac{d^2y}{dx^2} + c^2 y\frac{dy}{dx} + cy$

$\neq cD(y)$

So This is nonlinear. It's a second order equation.

b) $xy + \frac{dy}{dx} + \sin(x) = 0$

write as $xy + \frac{dy}{dx} = -\sin(x)$

or $D(y) = -\sin x$

where $D(y) = xy + \frac{dy}{dx}$ over

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But

$$D(y_1+y_2) = x(y_1+y_2) + \frac{d}{dx}(y_1+y_2)$$

$$= xy_1 + \frac{dy_1}{dx} + xy_2 + \frac{dy_2}{dx}$$

$$= D(y_1) + D(y_2) \checkmark$$

and $D(cy) = x(cy) + \frac{d}{dx}(cy) = c\,D(y) \checkmark$

So it's linear, But $-\sin x \neq 0$

So The equation is not homogeneous.

It's a <u>first</u> order differential equation.

c) $e^x \frac{d^3y}{dx^3} = y$

Write as $e^x \frac{d^3y}{dx^3} - y = 0$

or $D(y) = 0$ where $D(y) = e^x \frac{d^3y}{dx^3} - y$

$$D(y_1+y_2) = e^x\left(\frac{d^3}{dx^3}(y_1+y_2)\right) - (y_1+y_2)$$

$$= \left(e^x \frac{d^3y_1}{dx^3} - y_1\right) + e^x\left(\frac{d^3y_2}{dx^3} - y_2\right) \text{(arc)}$$

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$= D(y_1) + D(y_2)$ ✓

$D(cy) = e^x \frac{d^3(cy)}{dx^3} - cy$

$= c\left(e^x \frac{d^3y}{dx^3} - y\right) = c\,D(y)$ ✓

So it's linear

Also $D(y) = 0$ so it's homogonoous

It's <u>Third</u> order

d) $\frac{d^2y}{dx^2} = x^2 y + \frac{dy}{dx}$

write as $\frac{d^2y}{dx^2} - \frac{dy}{dx} - x^2 y = 0$

or $D(y) = 0$ where

$D(y) = \frac{d^2y}{dx^2} - \frac{dy}{dx} - x^2 y$.

Easy to see

$D(y_1 + y_2) = D(y_1) + D(y_2)$ ✓

and $D(cy) = c\,D(y)$ ✓ (over)

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So its Linear

$D(y) = \underline{\underline{0}}$ so its homogeneous

It's a second order differential equation.

3a) Show $u = e^{-x}$ solves $\frac{du}{dx} + u = 0$

$$\frac{du}{dx} = \frac{d}{dx} e^{-x} = -e^{-x}$$

$$+u \qquad + e^{-x}$$

$$= 0 \checkmark$$

b) Show $u = \frac{-x}{\log x}$ solves $x^2 \frac{du}{dx} = u^2 + xu$

$$\frac{du}{dx} = -\frac{d}{dx} \frac{x}{\log(x)} = -\left(\frac{\log(x) - x\frac{1}{x}}{(\log x)^2} \right)$$

$$= -\frac{1}{\log(x)} + \frac{1}{(\log(x))^2}$$

$$u^2 + xu = \left(\frac{x}{\log(x)}\right)^2 - x\frac{x}{\log(x)}$$

$$= x^2 \left(\frac{1}{(\log(x))^2} - \frac{1}{\log(x)} \right) \text{ (one)}$$

So $x^2 \frac{du}{dx} = x^2\left(-\frac{1}{\log(x)} + \frac{1}{(\log x)^2}\right)$

$= u^2 + x^2 = x^2\left(\frac{1}{(\log x)^2} - \frac{1}{\log(x)}\right)$ ✓

c) $u(x) = 2\arctan(e^x)$ solves $\frac{du}{dx} = \sin u$

$\frac{du}{dx} = 2\,\frac{1}{1+(e^x)^2}\cdot e^x$

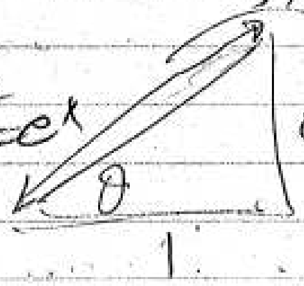
$\sin(u) = \sin(2\arctan(e^x))$

use $\sin(2\theta) = 2\sin(\theta)\cos\theta$

So
$\sin(2\arctan(e^x)) = 2\sin(\arctan(e^x))\cos(\arctan(e^x))$

$\sqrt{1+(e^x)^2}$

where $\tan(\theta) = e^x$



So $\sin(\theta) = \frac{e^x}{\sqrt{1+(e^x)^2}}$ $\quad \cos(\theta) = \frac{1}{\sqrt{1+(e^x)^2}}$

(over)

(8)

so $2\sin(\arctan(e^x))\cos(\arctan(e^x))$

$$= 2\,\frac{e^x}{\sqrt{1+(e^x)^2}} \cdot \frac{1}{\sqrt{1+(e^x)^2}}$$

$$= 2\,\frac{e^x}{1+(e^x)^2} = \frac{du}{dx} \text{ from above}$$

4a) $u(x) = xe^x$ solves $\frac{d^2u}{dx^2} - 2\frac{du}{dx} + u = 0$

$$\frac{du}{dx} = xe^x + e^x$$

$$\frac{d^2u}{dx^2} = xe^x + 2e^x$$

So $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y$

$$= (\cancel{xe^x} + \cancel{2e^x}) - 2(\cancel{xe^x} + \cancel{e^x}) + \cancel{xe^x}$$

$$= 0$$

b) $u = e^{e^x}$ solves $u\frac{d^2u}{dx^2} - \left(\frac{du}{dx}\right)^2 - u^2\log u = 0$

(over)

⑨

$$\frac{du}{dx} = \frac{d}{dx} e^{e^x} = e^{e^x} \cdot e^x$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(e^{e^x} \cdot e^x\right) = e^{e^x} e^x$$
$$+ e^{e^x}(e^x)^2$$

So $u\frac{d^2u}{dx^2} - \left(\frac{du}{dx}\right)^2 - u^2\log(u)$

$$= e^{e^x}\left(e^{e^x}e^x + e^{e^x}(e^x)^2\right)$$
$$- \left(e^{e^x}e^x\right)^2$$
$$- \left(e^{e^x}\right)^2 \log\left(e^{e^x}\right)$$
$$= \left(e^{e^x}\right)^2\left[\left(\cancel{e^x} + \cancel{(e^x)^2}\right) - \cancel{(e^x)^2} - e^x \cancel{\log(e)}\right]$$
$$= 0 \checkmark$$

c) $u = e^{x/2}$ solves $u\frac{d^2u}{dx^2} + \left(\frac{du}{dx}\right)^2 - \frac{1}{2}u^2 = 0$

$$\frac{du}{dx} = \frac{1}{2}e^{x/2}, \quad \frac{d^2u}{dx^2} = \frac{1}{4}e^{x/2}$$

(over)

(10)

So $u\frac{d^2u}{dx^2}+\left(\frac{du}{dx}\right)^2-\frac{1}{2}u^2$

$= e^{x/2}\,\frac{1}{4}e^{x/2} + \left(\frac{1}{2}e^{x/2}\right)^2 - \frac{1}{2}\left(e^{x/2}\right)^2$

$= e^x\left[\frac{1}{4}+\frac{1}{4}-\frac{1}{2}\right] = 0$ ✓

5) $\frac{d^2u}{dx^2}-u=0$ $\quad u_1=\sinh x$

$u_2=\cosh x$

$$\left.\begin{aligned}\frac{du_1}{dx}&=\cosh x\\ \frac{d^2u_1}{dx^2}&=\sinh(x)\end{aligned}\right\}\Rightarrow \frac{d^2u_1}{dx^2}-u_1=0$$

$$\left.\begin{aligned}\frac{du_2}{dx}&=\sinh x\\ \frac{d^2u_2}{dx^2}&=\cosh x\end{aligned}\right\}\Rightarrow \frac{d^2u_2}{dx^2}-u_2=0$$

Since the equation is linear and homogeneous

$u(x)=c_1\sinh x+c_2\cosh x$ solves

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$\frac{d^2u}{dx^2} - u = 0$ for any constants c_1 & c_2.

Solve IVPs

(a) $u(0) = 1 \quad u_x(0) = 2$

$1 = u(0) = c_1 \sinh 0 + c_2 \cosh 0 = c_2$

$2 = u_x(0) = c_1 \cosh 0 + c_2 \sinh 0 = c_1$

So $\boxed{u(x) = 2\sinh x + 1\cosh(x)}$

b) $u(0) = 2 \quad u_x(0) = 1$

$2 = u(0) = c_1 \sinh 0 + c_2 \cosh 0 = c_2$

$1 = u_x(0) = c_1 \cosh 0 + c_2 \sinh 0 = c_1$

So $\boxed{u(x) = 1\sinh x + 2\cosh x}$

Solve BVPs

(c) $u(0) = 0 \quad u(1) = 1$

$0 = u(0) = c_1 \sinh(0) + c_2 \cosh 0 = c_2$

$1 = u(1) = c_1 \sinh(1) + c_2 \cosh(1)$

But $c_2 = 0 \Rightarrow c_1 = \frac{1}{\sinh(1)} \Rightarrow$ (over)

(12)

$$\boxed{u(x) = \frac{1}{\sinh(1)} \sinh(x)}$$

d) $u(0) = 1 \qquad u(1) = 0$

$1 = u(0) = c_1 \sinh(0) + c_2 \cosh(0) = c_2$

$0 = u(1) = c_1 \sinh(1) + c_2 \cosh(1)$

$c_2 = 1 \Rightarrow \quad 0 = c_1 \sinh(1) + \cosh(1)$

$$\Rightarrow c_1 = -\frac{\cosh(1)}{\sinh(1)}$$

so

$$\boxed{u(x) = -\frac{\cosh(1)}{\sinh(1)} \sinh x + \cosh(x)}$$

e) $u_x(0) = 1 \qquad u_x(1) = 2$

$1 = u_x(0) = c_1 \cosh(0) + c_2 \sinh(0) = c_1$

$2 = u_x(1) = c_1 \cosh(1) + c_2 \sinh(1)$

$c_1 = 1 \Rightarrow \quad 2 = 1 \cosh(1) + c_2 \sinh(1)$

$c_2 = \frac{2 - \cosh(1)}{\sinh(1)}$

$$\boxed{u(x) = \sinh(x) + \frac{2 - \cosh(1)}{\sinh(1)} \cosh(x)}$$

(13)

6) $\frac{d^2u}{dx^2} - u = x$

$u_1 = \sinh x$

$u_2 = \cosh(x)$

$u_p = -x$

$\frac{d^2u_p}{dx^2} - u_p = 0 - (-x) = x$ ✓

So for this linear but inhomogeneous problem

$u(x) = c_1 \sinh x + c_2 \cosh x - x$

Solve the ODE for any c_1 & c_2.

Solve the IVPs

a) $u(0) = 1$ $u_x(0) = 2$

$1 = u(0) = c_1 \sinh(0) + c_2 \cosh 0 - 0 = c_2$

$2 = u_x(0) = c_1 \cosh(0) + c_2 \sinh(0) - 1 = c_1 - 1$

$c_2 = 1$ $c_1 = 3$

$$\boxed{u(x) = 3 \sinh x + \cosh x - x}$$

(14)

b) $u(0)=2 \quad u_x(0)=1$

$2=u(0)= c_1 \sinh(0) + c_2 \cosh(0) - 0 = c_2$

$1=u_x(0) = c_1 \cosh(0) + c_2 \sinh(0) - 1 = c_1 - 1$

$c_2 = 2 \quad c_1 = 2 \quad \boxed{u(x) = 2\sinh x + 2\cosh x - x}$

Solve the BVPs

c) $u(0)=0 \quad u(1)=1$

$0 = u(0) = c_1 \sinh(0) + c_2 \cosh(0) - 0 = c_2$

$1 = u(1) = c_1 \sinh(1) + c_2 \cosh(1) - 1$

But $c_2 = 0 \Rightarrow \quad 1 = c_1 \sinh(1) - 1$

$$c_1 = \frac{2}{\sinh(1)}$$

So $\boxed{u(x) = \frac{2}{\sinh(1)} \sinh(x) - x}$

d) $u(0)=1 \quad u(1)=0$

$1 = u(0) = c_1 \sinh(0) + c_2 \cosh(0) - 0 = c_2$

$0 = u(1) = c_1 \sinh(1) + c_2 \cosh(1) - 1 =$ (over)

(15)

But $C_2 = 1$ so $0 = C_1 \sinh(1) + \cosh(1) - 1$

$\Rightarrow \quad C_1 = \frac{1 - \cosh(1)}{\sinh(1)}$

so $\boxed{u(x) = \frac{1 - \cosh(1)}{\sinh(1)} \sinh(x) + \cosh(x) - x}$

5) $u_x(0) = 1 \quad u_x(1) = 2$

$1 = C_1 \cosh(0) + C_2 \sinh(0) - 1$

$2 = C_1 \cosh(1) + C_2 \sinh(1) - 1$

$2 = C_1$ so $2 = 2\cosh(1) + C_2 \sinh(1) - 1$

or $\frac{3 - 2\cosh(1)}{\sinh(1)} = C_2$

so $\boxed{u(x) = 2\sinh(x) + \frac{3 - 2\cosh(1)}{\sinh(1)} \cosh x - x}$

7) $u(x) = x\left(1 - \frac{2}{\log(x) + c}\right)$ solves

$2x^2 \frac{dy}{dx} - y^2 = x^2$

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a) Verify solution

$$\frac{du}{dx} = x\left(0 + \frac{2}{(\log|x|+c)^2}\cdot\frac{1}{x}\right)$$
$$+ 1\left(1 - \frac{2}{\log|x|+c}\right)$$
$$= \frac{2}{(\log|x|+c)^2} + \left(1 - \frac{2}{\log|x|+c}\right)$$

$$u^2 = x^2\left(1 - \frac{2}{\log|x|+c}\right)^2$$
$$= x^2\left(1 - \frac{4}{\log|x|+c} + \frac{4}{(\log|x|+c)^2}\right)$$

So

$$2x^2\frac{du}{dx} - u^2$$
$$= \cancel{\frac{4x^2}{(\log|x|+c)^2}} + 2x^2 - \cancel{\frac{4x^2}{\log|x|+c}}$$
$$- \left(x^2 - \cancel{\frac{4x^2}{\log|x|+c}} - \cancel{\frac{4x^2}{(\log x+c)^2}}\right) =$$

(over)

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$= 2x^2 - x^2 = x^2$ ✓

b) IVP with $u(1) = 0$

$0 = u(1) = 1\left(1 - \frac{2}{\log 1 + c}\right) = 1 - \frac{2}{c}$

So $\frac{2}{c} = 1 \Rightarrow c = 2$

$\Rightarrow \boxed{u(x) = x\left(1 - \frac{2}{\log x + 2}\right)}$ $x>0$

c) IVP with $u(1) = 1$

$1 = u(1) = 1 - \frac{2}{c} \Rightarrow \frac{1}{c} = 0$

Deal with This by setting $d = \frac{1}{c}$

$\Rightarrow u(x) = x\left(1 - \frac{2/c}{1/c \log|x| + 1}\right)$

$= x\left(1 - \frac{2d}{d \log|x| + 1}\right) = x$

8) $u(x) = -\log|x - c_1| + c_2$ solves

$\frac{d^2u}{dx^2} - \left(\frac{du}{dx}\right)^2 = 0$

(18)

a) Verify

$$\frac{du}{dx} = -\frac{1}{x-c_1} + 0$$

$$\frac{d^2y}{dx^2} = +\frac{1}{(x-c_1)^2}$$

So $\dfrac{d^2y}{dx^2} - \left(\dfrac{dy}{dy}\right)^2 = \dfrac{1}{(x-c_1)^2} - \left(-\dfrac{1}{x-c_1}\right)^2$

$$= 0 \checkmark$$

b) IVP with $u(0)=0$ & $u_x(0)=1$

$$0 = u(0) = -\log|-c_1| + c_2$$

$$1 = u_x(0) = -\frac{1}{0-c_1} \Rightarrow c_1 = 1$$

and so $c_2 = \log 1 = 0$

$\Rightarrow u(x) = -\log(|x-1|)$

But near $x=0$ $|x-1| = 1-x$

$\Rightarrow u(x) = -\log(1-x)$

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c) IVP with $u(0)=1 \quad u_x(0)=0$

This is a bit tricky. If you plug into given general solution you'll get

$$1 = u(0) = -\log|0-c_1| + c_2$$

$$0 = u_x(0) = -\frac{1}{0-c_1} \quad ???$$

let's solve as follows

$$c_2 = 1 + \log c_1 \quad \text{so} \quad \text{take } c_1 > 0$$

$$u(x) = -\log(c_1 - x) + \log c_1 + 1$$

$$= \log\left(\frac{c_1}{c_1 - x}\right) + 1$$

$$= \log\left(\frac{1}{1 - \frac{1}{c_1}x}\right) + 1$$

Now we can let $\frac{1}{c_1} \to 0$ (see ???)

to get

$$\boxed{u(x) = \log(1) + 1 = 1}$$