①

HW 9

1) $A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$

A has only one eval, $\lambda = 2$, and one dim espace spanned by $r = (1,-1)^T$.

a) Let g be any non-eigen vector. Earlier I took $g = (1,1)$. Here take $g = (1,0) \notin$ eigenspace. So

$$g = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; r = \underbrace{\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}}_{(A-\lambda I)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So $\boxed{S = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}}$,

b) $S^{-1} = \frac{1}{1}\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ (Used Cramer's rule)

So

$$e^{At} = S\, e^{\overbrace{\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}}^{J_{\lambda=2}} t}\, S^{-1}$$

But $e^{\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}t} = \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix} = e^{2t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

(2)

and so

$$e^{At} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} e^{2t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= e^{2t} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} t & -1+t \\ 1 & 1 \end{pmatrix}$$

$$= e^{2t} \begin{pmatrix} t+1 & t \\ -t & 1-t \end{pmatrix}$$

c) Compute

$$\frac{d}{dt} e^{At} = \frac{d}{dt}\left(e^{2t} \begin{pmatrix} t+1 & t \\ -t & 1-t \end{pmatrix} \right)$$

$$= e^{2t} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + 2e^{2t} \begin{pmatrix} t+1 & t \\ -t & 1-t \end{pmatrix}$$

$$= e^{2t} \begin{pmatrix} 1+2(t+1) & 1+2t \\ -1-2t & -1+2(1-t) \end{pmatrix}$$

(*) $$= e^{2t} \begin{pmatrix} 3+2t & 1+2t \\ -1-2t & 1-2t \end{pmatrix}$$ (over)

③

And also compute

$$Ae^{At} = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix} e^{2t} \begin{pmatrix} t+1 & t \\ -t & 1-t \end{pmatrix}$$

$$= e^{2t} \begin{pmatrix} 3(t+1)-t & 3t+(1-t) \\ -(t+1)-t & -t+(1-t) \end{pmatrix}$$

(**) $$= e^{2t} \begin{pmatrix} 3+2t & 2t+1 \\ -2t-1 & 1-2t \end{pmatrix}$$

See that (*) and (**) are the same.

2) $A = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}$

a) Compute evals and evecs

$$\det(A - \lambda I) = \det \begin{pmatrix} -1-\lambda & 4 \\ -1 & 3-\lambda \end{pmatrix}$$

$$= (\lambda+1)(\lambda-3)+4 = \lambda^2 - 2\lambda + 1 = 0$$

$$\Rightarrow \lambda = \frac{2 \pm \sqrt{4-4}}{2} = 1$$

(4)

$$[A-\lambda I] = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$ So only 1-dim espace: span $\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$.

i.e. Can't be diagonalized.

b) $$J_\lambda = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \lambda$$

c) Clearly $g = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \notin \text{span}\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$

$$r = (A-\lambda I)g = \begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

So $\boxed{S = \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}.}$ $\quad S^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}$

Check $S^{-1} A . S$

$$= \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = J_{\lambda=1} \checkmark$$

⑤

3) $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\frac{d}{dt}\vec{x} = A\vec{x} \Rightarrow \vec{x}(t) = e^{At}\vec{x}(0)$$

So using exercise 1 :

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{2t}\begin{pmatrix} 1+t & t \\ -t & 1-t \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= e^{2t}\begin{pmatrix} 1+t+2t \\ -t+(1-t)2 \end{pmatrix} = e^{2t}\begin{pmatrix} 1+3t \\ 2-3t \end{pmatrix}$$

That is
$$\boxed{\begin{aligned} x(t) &= e^{2t}(1+3t) \\ y(t) &= e^{2t}(2-3t) \end{aligned}}$$

4) $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\frac{d\vec{x}}{dt} = A\vec{x} \Rightarrow \vec{x}(t) = e^{At}\vec{x}(0)$$

so $\vec{x}(t) = S e^{J_\lambda t} S^{-1}\vec{x}(0)$ (over)

⑥ see exercise 2

$$\vec{X}(t) = \underbrace{\begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}}_{S} \underbrace{e^{t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}}_{e^{J_\lambda t}} \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}}_{S^{-1}} \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{\vec{X}(0)}$$

$$= e^{t}\begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} t & -1-2t \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= e^{t}\begin{pmatrix} -2t+1 & 2+4t-2 \\ -t & 1+2t \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \underbrace{e^{t}\begin{pmatrix} 1-2t & 4t \\ -t & 1+2t \end{pmatrix}}_{e^{At}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= e^{t}\begin{pmatrix} 1-2t+8t \\ -t+2+4t \end{pmatrix} = \boxed{e^{t}\begin{pmatrix} 1+6t \\ 2+3t \end{pmatrix}}$$

so
$$\boxed{\begin{aligned} x(t) &= e^{t}(1+6t) \\ y(t) &= e^{t}(2+3t) \end{aligned}}$$

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5) Recall, if A is 2×2 and <u>NOT</u> diagonalizable, there is an S such that

$$S^{-1} A S = J_\lambda \Rightarrow A = S J_\lambda S^{-1}$$

where $J_\lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$.

Therefore, for analytic function $f(x)$ we have

$$f(At) = S\, f(J_\lambda t)\, S^{-1}$$

(see notes top of page 2)

where

$$(*) \quad f(J_\lambda t) = \begin{pmatrix} f(\lambda t) & t f'(\lambda t) \\ 0 & f(\lambda t) \end{pmatrix}$$

From exercise 1

$$A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}, \quad J_\lambda = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$$

$$S = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

These are the ones I computed on #1

and so from (**)

$$\sin(J_\lambda t) = \begin{pmatrix} \sin(2t) & t\cos(2t) \\ 0 & \sin(2t) \end{pmatrix}$$

$\therefore$

$$\sin(At) = S \sin(J_\lambda t) S^{-1}$$

$$= \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \sin(2t) & t\cos(2t) \\ 0 & \sin(2t) \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} t\cos(2t) & -\sin(2t) + t\cos(2t) \\ \sin(2t) & \sin(2t) \end{pmatrix}$$

$$= \begin{pmatrix} t\cos(2t) + \sin(2t) & -\cancel{\sin(2t)} + t\cos 2t + \cancel{\sin 2t} \\ -t\cos 2t & \sin 2t - t\cos 2t \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} \sin 2t + t\cos 2t & t\cos 2t \\ -t\cos 2t & \sin(2t) - t\cos 2t \end{pmatrix}}$$

You'll get same answer no matter what S you use, e.g. $S = \begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix}$.