①

3321
2nd order Linear
Part I

1a) $\left(\frac{d}{dx}+I\right)\left(\frac{d}{dx}+2I\right)u$

$= \left(\frac{d}{dx}+I\right)\left(\frac{du}{dx}+2u\right)$

$= \frac{d}{dx}\left(\frac{du}{dx}+2u\right)+\left(\frac{du}{dx}+2u\right)$

$= \frac{d^2u}{dx^2}+2\frac{du}{dx}+\frac{du}{dx}+2u = \boxed{\frac{d^2u}{dx^2}+3\frac{du}{dx}+2u}$

OK. Now that I've verified the factorization, solve $\left(\frac{d}{dx}+I\right)\left(\frac{d}{dx}+2I\right)u = 0$

let $v = \left(\frac{d}{dx}+2I\right)u = \frac{du}{dx}+2u$ (*)

$\left(\frac{d}{dx}+I\right)v = 0 \Rightarrow \frac{dv}{dx}+v=0$

$v = c_1 e^{-x} \Leftarrow e^{-x}\frac{d}{dx}(e^{x}v) = 0$

Now use (*) to solve for u

$\frac{du}{dx}+2u = v = c_1 e^{-x}$ so $e^{-2x}\frac{d}{dx}(e^{2x}u) = c_1 e^{-x}$

(over)

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So $\frac{d}{dx}(e^{2x}u) = c_1 e^x \Rightarrow \underline{e^{2x}u} = \int c_1 e^x = \underline{c_1 e^x + c_2}$

solve for $\boxed{\begin{aligned} u &= e^{-2x}(c_1 e^x + c_2) \\ &= c_1 e^{-x} + c_2 e^{-2x} \end{aligned}}$

b) $\left(\frac{d}{dx} + I\right)\left(\frac{d}{dx} + I\right)u = \left(\frac{d}{dx} + I\right)\left(\frac{du}{dx} + u\right)$

$= \frac{d}{dx}\left(\frac{du}{dx} + u\right) + \left(\frac{du}{dx} + u\right) = \boxed{\frac{d^2u}{dx^2} + 2\frac{du}{dx} + u}$

Now that it's factored solve

$\left(\frac{d}{dx} + I\right)\underbrace{\boxed{\left(\frac{d}{dx} + I\right)u}}_{=v} = 0 \Rightarrow \frac{dv}{dx} + v = 0$

$\Downarrow$

$v = c_1 e^{-x} \Leftarrow e^{-x}\frac{d}{dx}(e^x v) = 0$

Finally, solve

$\left(\frac{du}{dx} + u\right) = v = c_1 e^{-x} \Rightarrow e^{-x}\frac{d}{dx}(e^x u) = c_1 e^{-x}$

$\Rightarrow \int \frac{d}{dx}(e^x u) = \int c_1 \Rightarrow e^x u = c_1 x + c_2 \Rightarrow \boxed{u = (c_1 x + c_2)e^{-x}}$

(2)

c) $\frac{d}{dx}\left(\frac{d}{dx}-\frac{1}{x}I\right)u = \frac{d}{dx}\left(\frac{du}{dx}-\frac{1}{x}u\right)$

$= \frac{d^2u}{dx^2} - \frac{d}{dx}\left(\frac{1}{x}u\right) = \boxed{\frac{d^2u}{dx^2} - \frac{1}{x}\frac{du}{dx} + \frac{1}{x^2}u}$

Now, solve homogeneous problem

$\frac{d}{dx}\underbrace{\left(\left(\frac{d}{dx}-\frac{1}{x}I\right)u\right)}_{=v} = 0 \Rightarrow \frac{dv}{dx} = 0 \Rightarrow v = c_1$

Solve for u: $\frac{du}{dx} - \frac{1}{x}u = v = c_1$

$\qquad //$

$e^{\int\frac{1}{x}}\frac{d}{dx}\left(e^{-\int\frac{1}{x}}u\right) = c_1$

But $e^{-\int\frac{1}{x}} = e^{-\log x} = e^{\log\frac{1}{x}} = \frac{1}{x}$

$\qquad //$

$x\frac{d}{dx}\left(\frac{1}{x}u\right) = c_1$

$\Rightarrow \int\frac{d}{dx}\left(\frac{1}{x}u\right) = \int\frac{c_1}{x} = c_1\log x + c_2$

$\qquad //$

$\frac{1}{x}u$

So $\boxed{u = x\,(c_1\log x + c_2)}$

④

d) $\left(\frac{d}{dx} + 2xI\right)\left(\frac{d}{dx}\right)u = \left(\frac{d}{dx}+2xI\right)\frac{du}{dx}$

$$= \boxed{\frac{d^2y}{dx^2} + 2x\frac{dy}{dx}}$$

Now, solve the homog problem

$$\left(\frac{d}{dx}+2xI\right)\left(\frac{du}{dx}\right) = 0 \Rightarrow \frac{dv}{dx} + 2xv = 0 \qquad \frac{du}{dx} = v$$

$$e^{\int 2x} = e^{x^2}$$

$$v = c_1 e^{-x^2} \Leftarrow e^{-x^2}\frac{d}{dx}\left(e^{x^2}v\right) = 0$$

And solve for u

$$\frac{du}{dx} = v = c_1 e^{-x^2} \Rightarrow \boxed{\begin{aligned} u &= \int c_1 e^{-x^2} + c_2 \\ &= c_1\,\mathrm{erf}(x) + c_2 \end{aligned}}$$

2a) To solve the inhomogeneous problem

$$\frac{d^2u}{dx^2} + 3\frac{du}{dx} + 2u = x$$

|| (From problem 1)

$$\left(\frac{d}{dx}+I\right)\left(\frac{d}{dx}+2I\right)u = x$$

(over)

(5)

let $v = \frac{dy}{dx} + 2y$ and solve

$$\frac{dv}{dx} + v = x \Rightarrow e^{-x}\frac{d}{dx}(e^{x}v) = x$$

$$\Rightarrow \int \frac{d}{dx}(e^{x}v) = \int x e^{x} \quad \left(\begin{smallmatrix}1 \text{ by} \\ \text{parts}\end{smallmatrix}\right) = (x-1)e^{x} + c_1$$

So $e^{x}v = (x-1)e^{x} + c_1 \Rightarrow v = (x-1) + c_1 e^{-x}$

Finally, solve for y

$$\frac{dy}{dx} + 2y = v = (x-1) + c_1 e^{-x}$$

$$\parallel$$

$$e^{-2x}\frac{d}{dx}(e^{2x}y) = (x-1) + c_1 e^{-x}$$

$$\int \frac{d}{dx}(e^{2x}y) = \int (x-1)e^{2x} + \int c_1 e^{x}$$

1 by Parts

$$e^{2x}y = \left(\tfrac{1}{2}(x-1) - \tfrac{1}{4}\right)e^{2x} + c_1 e^{x} + c_2$$

So $$\boxed{y = \left(\tfrac{1}{2}(x-1) - \tfrac{1}{4}\right) + c_1 e^{-x} + c_2 e^{-2x}}$$

(6) Again, use factorization from #1

b) $\left(\frac{d}{dx}+I\right)\left(\frac{d}{dx}+I\right)u = e^{-x}$

Let $v = \frac{du}{dx}+u$ so

$$\frac{dv}{dx}+v = e^{-x} \Rightarrow e^{-x}\frac{d}{dx}(e^{x}v) = e^{-x}$$

$$\Rightarrow \underbrace{\int \frac{d}{dx}(e^{x}v)}_{e^{x}v} = \int 1 = x + c_1$$

so $v = (x+c_1)e^{-x}$

Solve for u

$$\frac{du}{dx}+u = v = (x+c_1)e^{-x}$$

$$\frac{du}{dx}+u = e^{-x}\frac{d}{dx}(e^{x}u) \Rightarrow \int \frac{d}{dx}(e^{x}u) = \int x + c_1$$

$$e^{x}u = \int \frac{d}{dx}(e^{x}u) = \frac{x^2}{2}+c_1x+c_2$$

$$\boxed{u = \left(\frac{x^2}{2}+c_1x+c_2\right)e^{-x}}$$

(7)

c) $\left(\frac{d}{dx}\right)\underbrace{\left(\frac{du}{dx} - \frac{1}{x}u\right)}_{=v} = 1 \Rightarrow \frac{dv}{dx} = 1$

so $v = x + c_1$

$\frac{du}{dx} - \frac{1}{x}u = v = x + c_1$

$\overset{\parallel}{x\frac{d}{dx}\left(\frac{1}{x}u\right)} = x + c_1$ $\quad \underbrace{\int \frac{d}{dx}\left(\frac{1}{x}u\right)}_{=\frac{1}{x}u} = \int 1 + \frac{c_1}{x} = x + c_1 \log x + c_2$

$\Rightarrow \boxed{u = (c_1 \log x + c_2)x + x^2}$

d) $\left(\frac{d}{dx} + 2xI\right)\underbrace{\left(\frac{du}{dx}\right)}_{=v} = x \Rightarrow \frac{dv}{dx} + 2xv = x$

$\underbrace{\int \frac{d}{dx}(e^{x^2}v)}_{=e^{x^2}v} = \int x e^{x^2}$ substi

$v = \frac{1}{2} + c_1 e^{-x^2} \Leftarrow e^{x^2}v = \frac{1}{2}e^{x^2} + c_1$

$\frac{du}{dx} = v = \frac{1}{2} + c_1 e^{-x^2}$

$u = \int \frac{1}{2} + c_1 e^{-x^2} = \boxed{1/2\, x + c_1 \operatorname{erf}(x) + c_2}$

3a) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y$

$\quad = \boxed{\left(\dfrac{d}{dx} - 1I\right)\left(\dfrac{d}{dx} - 1I\right)y} \Leftarrow$

Char poly is

$$\begin{cases} r^2 - 2r + 1 = 0 \\ r = \dfrac{2 \pm \sqrt{4-4}}{2} \\ \quad = 1 \pm 0 \end{cases}$$

check $\left(\dfrac{d}{dx} - I\right)\left(\dfrac{dy}{dx} - y\right)$

$$= \frac{d}{dx}\left(\frac{dy}{dx} - y\right) - \left(\frac{dy}{dx} - y\right) = \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y \checkmark$$

b) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y$

$\quad = \boxed{\left(\dfrac{d}{dx} - I\right)\left(\dfrac{d}{dx} - 2I\right)y} \Leftarrow$

Char poly

$$\begin{cases} r^2 - 3r + 2 = 0 \\ r = \dfrac{3 \pm \sqrt{9 - 4 \cdot 2}}{2} \\ \quad = \dfrac{3}{2} \pm \dfrac{1}{2} = 1, 2 \end{cases}$$

check

$$= \left(\frac{d}{dx} - I\right)\left(\frac{dy}{dx} - 2y\right)$$

$$= \frac{d}{dx}\left(\frac{dy}{dx} - 2y\right) - \left(\frac{dy}{dx} - 2y\right) = \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y \checkmark$$

①

c) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y$

$\qquad\qquad\shortparallel$

$$\boxed{\left(\frac{d}{dx} - (1+i)I\right)\left(\frac{d}{dx} - (1-i)I\right)y} \Leftarrow \begin{cases} r^2 - 2r + 2 = 0 \\ r = \dfrac{2 \pm \sqrt{4 - 4\cdot 2}}{2} \\ \quad = 1 \pm \sqrt{-1} = 1 \pm i \end{cases}$$

check

$$\frac{d}{dx}\left(\frac{dy}{dx} - (1-i)y\right) - (1+i)\left(\frac{dy}{dx} - (1-i)y\right)$$

$$= \frac{d^2y}{dx^2} - (1-i)\frac{dy}{dx} - (1+i)\frac{dy}{dx} + \underbrace{(1+i)(1-i)}_{1+i-i+1}y$$

$$= \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y \quad \checkmark$$

d) $\dfrac{d^2y}{dx^2} - \dfrac{1}{x}\dfrac{dy}{dx} + \dfrac{1}{x^2}y$

$\qquad\qquad\shortparallel$

$$\boxed{\left(\frac{d}{dx} - \frac{(1-1)}{x}I\right)\left(\frac{d}{dx} - \frac{1}{x}I\right)y}$$

C-E char poly is

$r(r-1) - r + 1 = 0$

$r^2 - 2r + 1 = 0$

$r = \dfrac{2 \pm \sqrt{4-4}}{2} = 1 \pm 0$

Check

$$= \frac{d}{dx}\left(\frac{dy}{dx} - \frac{1}{x}y\right)$$

$$= \frac{d^2y}{dx^2} - \frac{d}{dx}\left(\frac{1}{x}y\right) = \frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + \frac{1}{x^2}y \quad \checkmark$$

e) $$\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + \frac{2}{x^2}y$$

CE char poly

$$r(r-1) - 2r + 2 = 0$$

$$r^2 - 3r + 2 = 0$$

$$r = \frac{3 \pm \sqrt{9-8}}{2} = \frac{3}{2} \pm \frac{1}{2}$$

$$= 1, 2$$

$$\parallel$$

$$\boxed{① = \left(\frac{d}{dx} - \frac{(1-1)}{x}I\right)\left(\frac{d}{dy} - \frac{2}{x}I\right)y}$$

or BTW could have switched around roots

$$\boxed{② = \left(\frac{d}{dx} - \frac{2-1}{x}I\right)\left(\frac{d}{dx} - \frac{1}{x}I\right)y}$$

I'll show you **both** are right.

$$① = \left(\frac{d}{dx}\right)\left(\frac{dy}{dx} - \frac{2}{x}y\right) = \frac{d^2y}{dx^2} - \frac{d}{dx}\left(\frac{2}{x}y\right) = \frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + \frac{2}{x^2}y \quad ✓$$

$$② = \left(\frac{d}{dx} - \frac{1}{x}I\right)\left(\frac{dy}{dx} - \frac{1}{x}y\right) = \frac{d}{dx}\left(\frac{dy}{dx} - \frac{1}{x}y\right) - \frac{1}{x}\left(\frac{dy}{dx} - \frac{1}{x}y\right)$$

$$= \frac{d^2y}{dx^2} - \frac{d}{dx}\left(\frac{1}{x}y\right) - \frac{1}{x}\frac{dy}{dx} + \frac{1}{x^2}y$$

$$= \frac{d^2y}{dx^2} - \frac{1}{x}\cancel{\frac{dy}{dx}} + \cancel{\frac{1}{x^2}y} - \frac{1}{x}\cancel{\frac{dy}{dx}} + \cancel{\frac{1}{x^2}y}$$

$$= \frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dy} + \frac{2}{x^2}y \quad ✓$$

either factorization will work just fine.

⑪

f) $\frac{d^2u}{dx^2} - \frac{1}{x}\frac{du}{dx} + \frac{2}{x^2}u$

$r(r-1) - r + 2 = 0$

$r^2 - 2r + 2 = 0$

$r = \frac{2 \pm \sqrt{4 - 4 \cdot 2}}{2}$

$= 1 \pm i$

$\parallel$

$$\left(\frac{d}{dx} - \frac{(1+i-1)}{x}I\right)\left(\frac{d}{dx} - \frac{(1-i)}{x}I\right)u$$

But also could have used

$$\left(\frac{d}{dx} - \frac{(1-i-1)}{x}I\right)\left(\frac{d}{dx} - \frac{1+i}{x}I\right)u$$

I'll check the first one only.

$$\frac{d}{dx}\left(\frac{du}{dx} - \frac{(1-i)}{x}u\right) - \frac{i}{x}\left(\frac{du}{dx} - \frac{1-i}{x}u\right)$$

$$= \frac{d^2u}{dx^2} - (1-i)\frac{d}{dx}\left(\frac{1}{x}u\right) - \frac{i}{x}\frac{du}{dx} + \frac{i(1-i)}{x^2}u$$

$$= \frac{d^2u}{dx^2} - (1\cancel{-i})\left(\frac{1}{x}\frac{du}{dx} - \frac{1}{x^2}u\right) - \frac{i}{x}\frac{du}{dx} + \frac{i(1-i)}{x^2}u$$

$$= \frac{d^2u}{dx^2} - \left((1\cancel{-i}) \cancel{+i}\right)\frac{1}{x}\frac{du}{dx} + \frac{(1\cancel{-i}) + (\cancel{i}+1)}{x^2}u$$

$$= \frac{d^2u}{dx^2} - \frac{1}{x}\frac{du}{dx} + \frac{2}{x^2}u \quad \checkmark$$

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To solve the homogeneous constant coefficient problem you need only find roots of char polynomial and use formulae on page 5 (memorize these) for the 3 cases

4a) $\frac{d^2u}{dx^2} - u = 0 \Rightarrow r^2 - 1 = 0 \Rightarrow r = \pm 1$

real and distinct

so $u(x) = e^{0x}(c_1 \cosh(1x) + c_2 \sinh(1x))$

$= \boxed{c_1 \cosh(x) + c_2 \sinh(x)}$

b) $\frac{d^2u}{dx^2} - \frac{du}{dx} - 2u = 0 \Rightarrow r^2 - r - 2 = 0$

$r = \frac{1 \pm \sqrt{1 + 4 \cdot 2}}{2}$

$$\boxed{u(x) = e^{\frac{x}{2}}\left(c_1 \cosh\left(\tfrac{3}{2}x\right) + c_2 \sinh\left(3/2x\right)\right)} \Leftarrow r = \frac{1}{2} \pm \frac{3}{2}$$

c) $\frac{d^2u}{dx^2} = 0 \Rightarrow r^2 = 0 \Rightarrow r = 0 \pm 0$

$\boxed{u(x) = e^{0x}(c_1 x + c_2)}$

only one root!

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d) $\frac{d^2u}{dx^2} + u = 0 \Rightarrow r^2 + 1 = 0$ $r = 0 \pm i$

complex

$u(x) = e^{0x}(c_1 \cos x + c_2 \sin x)$

$= \boxed{c_1 \cos x + c_2 \sin x}$

e) $\frac{d^2u}{dx^2} - 2\frac{dy}{dx} + 2u = 0$ $r^2 - 2r + 2 = 0$

$r = \frac{2 \pm \sqrt{4 - 4 \cdot 2}}{2}$

$= 1 \pm i$

complex

$\boxed{u(x) = e^{x}(c_1 \cos x + c_2 \sin x)}$

f) $\frac{d^2u}{dx^2} + 2\frac{dy}{dx} + u = 0 \Rightarrow r^2 + 2r + 1 = 0$

$r = \frac{-2 \pm \sqrt{4 - 4 \cdot 1}}{2}$

$= -1 \pm 0$

one distinct root

$\boxed{u(x) = e^{-x}(c_1 x + c_2)}$

(IVPs)

5a) $\frac{d^2u}{dx^2}=0$, $u(0)=1$ $u'(0)=2$

From 4c general soln is

$$u(x)=c_1x+c_2$$

$$1=u(0)=c_1\cdot 0+c_2 \Rightarrow c_2=1$$

$$2=u'(0)=c_1 \Rightarrow c_1=2$$

$$\boxed{u(x)=2x+1}$$

b) $\frac{d^2u}{dx^2}-\frac{dy}{dx}+2u=0$, $u(0)=1$ $u'(0)=2$

Get the general solution

$$r^2-r+2=0 \Rightarrow r=\frac{1\pm\sqrt{1-4\cdot 2}}{2}=\frac{1\pm\sqrt{7}\,i}{2}$$

$$u(x)=e^{\frac{x}{2}}\left(c_1\cos\left(\tfrac{\sqrt7}{2}x\right)+c_2\sin\left(\tfrac{\sqrt7}{2}x\right)\right) \Leftarrow$$

Now solve for c_1 & c_2:

$$1=u(0)=e^0(c_1\cos(0)+c_2\sin(0))=c_1 \Rightarrow \boxed{c_1=1}$$

$$2=u'(0)=e^0\left(-\tfrac{\sqrt7}{2}c_1\sin(0)+\tfrac{\sqrt7}{2}c_2\cos(0)\right)+\tfrac12 e^0(c_1\cos(0)+c_2\sin(0))=\tfrac{\sqrt7}{2}c_2+\tfrac12 c_1$$

So $u(x)=e^{x/2}\left(\cos\left(\frac{\sqrt7}{2}x\right)+\frac{3}{\sqrt7}\sin\left(\frac{\sqrt7}{2}x\right)\right)$

$$c_2=\frac{2-1/2}{\sqrt7/2}=\frac{3}{\sqrt7}$$

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BVPs

6a) $\frac{d^2u}{dx^2} - u = 0 \quad u(0) = 1, \; u(1) = 2$

General solution is (see 4a)

$u(x) = c_1 \cosh(x) + c_2 \sinh(x)$

$1 = u(0) = c_1 \cosh(0) + c_2 \sinh(0) = c_1 \Rightarrow \boxed{c_1 = 1}$

$2 = u(1) = c_1 \cosh(1) + c_2 \sinh(1)$

$\Rightarrow \boxed{c_2 = \frac{2 - 1 \cdot \cosh(1)}{\sinh(1)}}$

$$u(x) = \cosh(x) + \frac{2 - \cosh(1)}{\sinh(1)} \sinh(x)$$

b) This time Neumann BC is $u'(0) = 1$, $u'(1) = 2$

$1 = u'(0) = c_1 \sinh(0) + c_2 \cosh(0)$

$= c_2 \Rightarrow \boxed{c_2 = 1}$

$2 = u'(1) = c_1 \sinh(1) + c_2 \cosh(1)$

$c_1 = \frac{2 - 1 \cdot \cosh(1)}{\sinh(1)}$

so
$$u(x) = \frac{2 - \cosh(1)}{\sinh(1)} \cosh(x) + \sinh(x)$$

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7) ~~Homogeneous~~ C-E general solution

(See page 6)

a) $\frac{d^2u}{dx^2} - \frac{3}{x}\frac{du}{dx} + \frac{3}{x^2}u = 0 \Rightarrow r(r-1) - 3r + 3 = 0$

$r^2 - 4r + 3 = 0$

$r = \frac{4 \pm \sqrt{16 - 4 \cdot 3}}{2}$

$= 2 \pm 1 \ (= 1, 3)$

$\boxed{u(x) = c_1 x + c_2 x^3}$

b) $\frac{d^2u}{dx^2} - \frac{3}{x}\frac{du}{dx} + \frac{4}{x^2}u = 0 \Rightarrow r(r-1) - 3r + 4 = 0$

$r^2 - 4r + 4 = 0$

$r = \frac{4 \pm \sqrt{16 - 4 \cdot 4}}{2}$

$= 2 \pm 0$

only one root

$u(x) = (c_1 \log(x) + c_2) x^2$

8a) General solution to $\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} = 0$

$r(r-1) + r = 0$

$r = 0 \pm 0$

is $u(x) = c_1 \log(x) + c_2$

(over)

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BC1

$a_0 = u(1) = c_1 \log(1) + c_2 \Rightarrow \boxed{c_2 = a_0}$

$b_0 = u(2) = c_1 \log(2) + c_2 \Rightarrow \frac{b_0 - a_0}{\log(2)} = c_1$

so $\boxed{u(x) = \left(\frac{b_0 - a_0}{\log(2)}\right)\log(x) + a_0}$

b) General solution to

$$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} - \frac{n^2}{x^2}u = 0$$

$n = 1, 2, \ldots$

$r(r-1) + r - n^2 = 0$

$r - n^2 = 0$

$r = \pm n$

$u(x) = c_1 x^n + c_2 x^{-n}$

B, C1

$a_n = u(1) = c_1 + c_2$

$b_n = u(2) = c_1 2^n + c_2 2^{-n}$

coupled equation for c_1 & c_2

mult by 2^n

$a_n = c_1 + c_2$

$2^n b_n = c_1 (2^n)^2 + c_2$

$2^n b_n - a_n = c_1 ((2^n)^2 - 1) \Rightarrow c_1 = \frac{2^n b_n - a_n}{(2^n)^2 - 1}$

(over)

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$$a_n = c_1 + c_2 \Rightarrow c_2 = a_n - c_1$$
$$= a_n - \left[\frac{2^n b_n - a_n}{(2^n)^2 - 1}\right]$$

Those look "prettier" if I do

$$c_1 = \frac{2^n b_n - a_n}{(2^n)^2 - 1} = \frac{2^n(b_n - 2^{-n} a_n)}{2^n(2^n - 2^{-n})} = \frac{b_n - 2^{-n} a_n}{2^n - 2^{-n}}$$

$$c_2 = \frac{a_n(2^n - \cancel{2^{-n}}) - (b_n - \cancel{2^{-n} a_n})}{2^n - 2^{-n}} = \frac{2^n a_n - b_n}{2^n - 2^{-n}}$$

So

$$u(x) = \frac{1}{2^n - 2^{-n}}\left((b_n - 2^{-n} a_n) x^n + (2^n a_n - b_n) x^{-n}\right)$$

$n = 1, 2, \ldots$