1

Second Order Linear Part II

1) Method of guessing

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 0$$

homog solution;

$$r^2 + 2r - 3 = 0$$

$$r = \frac{-2 \pm \sqrt{4 + 4 \cdot 3}}{2}$$

$$= -1 \pm \sqrt{4} = -1 \pm 2 \qquad -3, 1$$

$$y_h(x) = e^{-x}\left(c_1 \cosh(2x) + c_2 \sinh(2x)\right)$$

$$\text{-or-} \quad = c_1 e^{-3x} + c_2 e^{x}$$

Now solve $\quad \frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = f(x)$

a) $f(x) = x^2 + 1$

Try $y_p(x) = \alpha_2 x^2 + \alpha_1 x + \alpha_0$

This will work since $r=1$ does not solve homog equation.

$$\frac{dy_p}{dx} = 2\alpha_2 x + \alpha_1, \quad \frac{d^2y_p}{dx^2} = 2\alpha_2$$

(over)

②

Plug in $\qquad\qquad = x^2+1$

$$(2\alpha_2) + 2(2\alpha_2 x + \alpha_1) - 3(\alpha_2 x^2 + \alpha_1 x + \alpha_0)$$

$$= -3\alpha_2 x^2 + (4\alpha_2 - 3\alpha_1)x + \begin{pmatrix} 2\alpha_2 + 2\alpha_1 \\ -3\alpha_0 \end{pmatrix}$$

So need

$$\begin{aligned} -3\alpha_2 &= 1 \\ 4\alpha_2 - 3\alpha_1 &= 0 \\ 2\alpha_2 + 2\alpha_1 - 3\alpha_0 &= 1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \alpha_2 &= -1/3 \\ \alpha_1 &= -4/9 \\ \alpha_0 &= \tfrac{1}{3}\left(-\tfrac{2}{3} - \tfrac{8}{9} - 1\right) \\ &= -\tfrac{23}{27} \end{aligned}$$

So $u_p(x) = -1/3\, x^2 - 4/9\, x - 23/27$

and the general solution is

$$\boxed{\begin{aligned} u(x) = e^{-x}&\left(c_1 \cosh(2x) + c_2 \sinh(2x)\right) \\ &- 1/3\, x^2 - 4/9\, x - 23/27 \end{aligned}}$$

b) $f(x) = \sin(x)$

Try $u_p(x) = \alpha \sin x + \beta \cos x$

will work since neither $\sin(x)$ or $\cos x$ solve homog. problem

(3)

$$\frac{du_p}{dy} = \alpha\cos x - \beta\sin x$$

$$\frac{d^2u_p}{dx^2} = -\alpha\sin x - \beta\cos x$$

$$\frac{d^2u_p}{dx^2} + 2\frac{du_p}{dx} - 3u_p = \sin x$$

$$(-\cancel{\alpha}\sin x - \cancel{\beta}\cos x) + 2(\cancel{\alpha}\cos x - \cancel{\beta}\sin x)$$
$$-3(\alpha\sin x + \beta\cos x)$$

$$= (-\alpha - 2\beta - 3\alpha)\sin x$$
$$+ (-\beta + 2\alpha - 3\beta)\cos x$$

so need $-4\alpha - 2\beta = 1$

$2\alpha - 4\beta = 0$

$\Rightarrow \alpha = -2/10 \quad \beta = -\frac{1}{10}$

$$u_p = -\frac{2}{10}\sin x - \frac{1}{10}\cos x$$

$$\boxed{u(x) = e^{-x}(c_1\cosh(2x) + c_2\sinh(2x)) - 1/5\sin x - 1/10\cos x}$$

④

c) $f(x) = e^x$

$u_p(x) = \alpha e^x$ won't work bc, e^x solves homog problem

Instead

$u_p(x) = \alpha x e^x$ will work,

$\frac{du_p}{dx} = \alpha(xe^x + e^x)$, $\frac{d^2 u_p}{dx^2} = \alpha(xe^x + 2e^x)$

$\frac{d^2 u_p}{dx^2} + 2\frac{du_p}{dx} - 3u_p = e^x$

$\alpha(\cancel{xe^x} + 2e^x) + 2\alpha(\cancel{xe^x} + e^x) - 3\alpha \cancel{xe^x}$

$= \alpha(2+2)e^x \Rightarrow \alpha = \frac{1}{4}$

$u_p(x) = \frac{1}{4} x e^x$

and the general solution is

$$\boxed{u(x) = e^{-x}(c_1 \cosh 2x + c_2 \sinh 2x) + \frac{1}{4} x e^x}$$

5

d) $f(x) = x^2 + 1 + 5e^x$

$\mathcal{L}(u) = \frac{d^2u}{dx^2} + 2\frac{du}{dx} - 3u$

From part a: $\mathcal{L}(-1/3x^2 - 4/9x - 23/27) = x^2 + 1$

From c: $\left(\mathcal{L}(\frac{1}{4}xe^x) = e^x\right) \cdot 5$

So $u_p = (-1/3x^2 - 4/9x - 23/27)$
$+ 5(\frac{1}{4}xe^x)$

and the general soln is

$$\boxed{\begin{aligned} u &= e^{-x}(c_1 \cosh 2x + c_2 \sinh 2x) \\ &\quad -1/3x^2 - 4/9x - 23/27 + \frac{5}{4}xe^x \end{aligned}}$$

2 Homog solution first:

$\frac{d^2u}{dx^2} + 4u = 0$

$r^2 + 4 = 0 \qquad r = \pm 2i$

$u_h(x) = c_1 \cos(2x) + c_2 \sin(2x)$

⑥

a) $f(x) = 2x + 1$

Try $u_p = \alpha_1 x + \alpha_0$ will work,

$$\frac{du_p}{dx} = \alpha_1 \qquad \frac{d^2 u_p}{dx^2} = 0$$

$$\frac{d^2 u_p}{dx^2} + 4u_p = 2x + 1$$

$$0 + 4(\alpha_1 x + \alpha_0) = 4\alpha_1 x + 4\alpha_0$$

so $4\alpha_1 = 2$, $4\alpha_0 = 1$ $\Rightarrow$ $\alpha_1 = 1/2$, $\alpha_0 = 1/4$

$$u_p(x) = \frac{1}{2}x + \frac{1}{4}$$

and the general soln is

$$u(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{2}x + \frac{1}{4}$$

b) $f(x) = \sin x$

$u_p(x) = \alpha \sin x + \beta \cos x$ ← will work b.c. neither $\sin x$ or $\cos x$ solve homog eqn,

$$\frac{du_p}{dx} = \alpha \cos x - \beta \sin x$$

$$\frac{d^2 u_p}{dx^2} = -\alpha \sin x - \beta \cos x \quad \text{(over)}$$

(7)

$$\frac{d^2u_p}{dx^2} + 4u_p = \sin x$$

$\Downarrow$

$$(-\alpha \sin x - \beta \cos x) + 4(\alpha \sin x + \beta \cos x)$$

$$= 3\alpha \sin x + 3\beta \cos x \Rightarrow \begin{array}{l} \alpha = 1/3 \\ \beta = 0 \end{array}$$

$u_p = 1/3 \sin x$

and the general solution is

$$\boxed{u(x) = C_1 \cos 2x + C_2 \sin 2x + 1/3 \sin x}$$

c) $f(x) = \sin 2x$

$u_p = \alpha \sin 2x + \beta \cos 2x$ won't work b.c. $\sin 2x$ & $\cos 2x$ solve homog problem

instead try

$u_p(x) = x(\alpha \sin 2x + \beta \cos 2x)$ ← will work

$$\frac{du_p}{dx} = x \cdot (2\alpha \cos 2x - 2\beta \sin 2x) + \alpha \sin 2x + \beta \cos 2x$$

(over)

$$\frac{d^2u_p}{dx^2} = x(-4\alpha \sin 2x - 4\beta \cos 2x)$$
$$2\alpha \cos 2x - 2\beta \sin 2x + 2\alpha \cos 2x - 2\beta \sin 2x$$

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$$\frac{d^2 y_p}{dx^2} + 4 \cdot y_p = \sin 2x$$

$$\parallel$$

$$-4x\left(\alpha \cancel{\sin 2x} + \beta \cancel{\cos 2x}\right)$$

$$4\alpha \cos 2x - 4\beta \sin 2x$$

$$+4\left(x\left(\alpha \cancel{\sin 2x} + \beta \cancel{\cos 2x}\right)\right)$$

$$= 4\alpha \cos 2x - 4\beta \sin 2x$$

so $4\alpha = 0$ $\Rightarrow$ $\alpha = 0$

$-4\beta = 1$ $\qquad$ $\beta = -\frac{1}{4}$

$$y_p(x) = x\left(-\frac{1}{4}\cos 2x\right)$$

and the general soln is

$$\boxed{y(x) = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4} x \cos 2x}$$

d) $f(x) = 2\sin x + 3\sin 2x$

$\mathcal{L}(y) = \frac{d^2 y}{dx^2} + 4y$ $\qquad$ (once)

(9)

From b

2 $\mathcal{L}(1/3 \sin x) = 2 \sin x$

From c

3 $\mathcal{L}(-\frac{1}{4} x \cos 2x) = 3 \sin 2x$

So $u_p = \frac{2}{3} \sin x - \frac{3}{4} x \cos 2x$

solves $\mathcal{L}(u_p) = 2 \sin x + 3 \sin 2x$

and the general solution is

$$\boxed{u(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{2}{3} \sin x - \frac{3}{4} x \cos 2x}$$

3 Homog solution for $\frac{d^2u}{dx^2} - 9u = 0$

$r^2 - 9 = 0 \Rightarrow r = 0 \pm 3$

$$\boxed{u_h(x) = c_1 \cosh 3x + c_2 \sinh 3x}$$

-OR- $c_1 e^{3x} + c_2 e^{-3x}$

a) $f(x) = x^2$

$u_p = \alpha_2 x^2 + \alpha_1 x + \alpha_0$ will work.

(over)

(10)

$$\frac{du_p}{dx} = 2\alpha_2 x + \alpha_1 \qquad \frac{d^2u_p}{dy^2} = 2\alpha_2$$

so $\dfrac{d^2u_p}{dx^2} - 9u_p = x^2$

$$(2\alpha_2) - 9(\alpha_2 x^2 + \alpha_1 x + \alpha_0)$$

$$= -9\alpha_2 x^2 - 9\alpha_1 x + 2\alpha_2 - 9\alpha_0$$

$$\left.\begin{aligned} -9\alpha_2 &= 1 \\ -9\alpha_1 &= 0 \\ 2\alpha_2 - 9\alpha_0 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} \alpha_2 &= -\tfrac{1}{9} \\ \alpha_1 &= 0 \\ \alpha_0 &= -2/81 \end{aligned}$$

$$u_p = -\frac{1}{9}x^2 - \frac{2}{81}$$

and the general solution is

$$\boxed{u(x) = c_1 \cosh 3x + c_2 \sinh 3x - \frac{1}{9}x^2 - \frac{2}{81}}$$

(11)

b) $f(x) = \sin 3x$

$y_p = \alpha \sin 3x + \beta \cos 3x$ ← will work

$\frac{dy_p}{dx} = 3\alpha \cos 3x - 3\beta \sin 3x$

$\frac{d^2y_p}{dx^2} = -9\alpha \sin 3x - 9\beta \cos 3x$

$\frac{d^2y_p}{dx^2} - 9y_p = \sin 3x$

$-9\alpha \sin 3x - 9\beta \cos 3x - 9(\alpha \sin 3x + \beta \cos 3x)$

$= -18\alpha \sin 3x - 18\beta \cos 3x$

$\Rightarrow \alpha = -\frac{1}{18} \quad \beta = 0$

$y_p = -\frac{1}{18} \sin 3x + 0$

and the general solution is

$y(x) = c_1 \cosh 3x + c_2 \sinh 3x - \frac{1}{18} \sin(3x)$

(12)

e) $f(x) = e^{-3x}$

$u_p = \alpha e^{-3x}$ won't work b.c. e^{-3x} solves homog problem.

But

$u_p = x\alpha e^{-3x}$ will work.

$$\frac{du_p}{dx} = \alpha\left(-3xe^{-3x} + e^{-3x}\right)$$

$$\frac{d^2u_p}{dx^2} = \alpha\left(+9xe^{-3x} - 6e^{-3x}\right)$$

so $$\frac{d^2u_p}{dx^2} - 9u_p = e^{-3x}$$

$$\alpha\left[9\cancel{xe^{-3x}} - 6e^{-3x}\right] - 9\alpha \cancel{xe^{-3x}}$$

$$= -6\alpha e^{-3x} \Rightarrow \alpha = -\frac{1}{6}$$

$u_p = -\frac{1}{6}xe^{-3x}$ and the general solution is

$$\boxed{u(x) = c_1\cosh 3x + c_2\sinh 3x - \frac{1}{6}xe^{-3x}}$$

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$$\mathcal{L}(u) = \frac{d^2u}{dx^2} - 9u$$

d) $f(x) = 9x^2 + 10e^{-3x}$

From a) $9\,\mathcal{L}\left(-\frac{1}{9}x^2 - \frac{2}{81}\right) = 9x^2$

From b) $10\,\mathcal{L}\left(-\frac{1}{6}xe^{-3x}\right) = 10\,e^{-3x}$

so $u_p = 9\left(-\frac{1}{9}x^2 - \frac{2}{81}\right) - \frac{10}{6}xe^{-3x}$

and the general solution is

$$\boxed{u(x) = c_1\cosh 3x + c_2\sinh 3x - x^2 - \frac{2}{9} - \frac{5}{3}xe^{-3x}}$$

4) $\mathcal{L}(u) \equiv \frac{d^2u}{dx^2} - 2\frac{du}{dx} + u$

Solve the homog problem

$\mathcal{L}(u) = 0 \qquad r^2 - 2r + 1 = 0$

$$r = \frac{2 \pm \sqrt{4-4}}{2}$$

$$= 1 \pm 0$$

so $u_h = e^x(c_1 + c_2x)$ (over)

(14)

a) $f(x) = 3x^2 + 4$

$u_p = \alpha_2 x^2 + \alpha_1 x + \alpha_0$ ← This'll will work B.C $u = 1$ does not solve homog problem.

$\frac{du_p}{dx} = 2\alpha_2 x + \alpha_1$

$\frac{d^2u_p}{dx^2} = 2\alpha_2$

so $\frac{d^2u_p}{dx^2} - 2\frac{du_p}{dx} + u_p = 3x^2 + 4$

$$= (2\alpha_2) - 2(2\alpha_2 x + \alpha_1) + (\alpha_2 x^2 + \alpha_1 x + \alpha_0)$$

$$= \alpha_2 x^2 + (\alpha_1 - 4\alpha_2)x + (\alpha_0 - 2\alpha_1 + 2\alpha_2)$$

So need

$\alpha_2 = 3$

$\alpha_1 - 4\alpha_2 = 0$

$\alpha_0 - 2\alpha_1 + 2\alpha_2 = 4$

$\Rightarrow$

$\alpha_2 = 3$

$\alpha_1 = 12$

$\alpha_0 = 24 - 6 + 4$

$= 22$

(15)

$u_p = 3x^2 + 12x + 22$

and the general soln is

$$\boxed{u(x) = e^x(c_1 + c_2 x) + 3x^2 + 12x + 22}$$

b) $f(x) = e^x$

$u_p = \alpha e^x$ won't work b.c. e^x solves homog problem.

$u_p = x\alpha e^x$ won't work b.c. xe^x solves the homog problem.

But

$u_p = x^2 \alpha e^x$ will work.

$$\frac{du_p}{dx} = \alpha\left(x^2 e^x + 2x e^x\right)$$

$$\frac{d^2u_p}{dx^2} = \alpha\left(x^2 + 4x e^x + 2e^x\right)$$

$$\frac{d^2u_p}{dx^2} - 2\frac{du_p}{dx} + u_p = e^x$$

$$\alpha\left[(\cancel{x^2} + \cancel{4x} + 2) - 2(\cancel{x^2} + \cancel{2x}) + \cancel{x^2}\right]e^x$$ (over)

(16)

$= 2\alpha \Rightarrow \alpha = 1/2$

$u_p = 1/2 x^2 e^x$

and the general soln is

$$\boxed{u(x) = e^x(c_1 + c_2 x) + \frac{1}{2}x^2 e^x}$$

c) $f(x) = e^{2x}$

$u_p = \alpha e^{2x}$ will work,

$\frac{du_p}{dx} = 2\alpha e^{2x}$ $\frac{d^2u_p}{dx^2} = 4\alpha e^{2x}$

so $\frac{d^2u_p}{dx^2} - 2\frac{du_p}{dx} + u_p = e^{2x}$

$\parallel$

$4\alpha e^{2x} - 2\, 2\alpha e^{2x} + \alpha e^{2x}$

$\parallel$

$\alpha e^{2x} \Rightarrow \alpha = 1$

$u_p = e^{2x}$ and gen soln

is $\boxed{u(x) = e^x(c_1 + c_2 x) + e^{2x}}$

(17)

d) $\mathcal{L}(u) = \frac{d^2u}{dx^2} - 2\frac{du}{dx} + u$

From a $\quad 2\,\mathcal{L}(3x^2+12x+22) = 2(3x^2+4)$

From b $\quad 5\,\mathcal{L}(\tfrac{1}{2}x^2e^x) = 5(e^x)$

From c $\quad 6\,\mathcal{L}(e^{2x}) = 6(e^{2x})$

So $u_p = 2(3x^2+12x+22)$
$\quad + 5(\tfrac{1}{2}x^2e^x)$
$\quad + 6(e^{2x})$

Satisfies

$$\mathcal{L}(u_p) = 2(3x^2+4) + 5e^x + 6e^{2x}$$

So the general soln to $\mathcal{L}(u) =$
$6x^2+8+5e^x+6e^{2x}$

is

$$u(x) = e^x(c_1+c_2x) + 6x^2+24x+44+\frac{5}{2}x^2e^x+6e^{2x}$$

(18)

5 $\mathcal{L}(u) = \frac{d^2u}{dx^2}$

Homog soln $\frac{d^2u}{dx^2} = 0$

$r^2 = 0$

$\Leftarrow r = 0 \; 2 = 0$

$u_h = e^{0x}(c_1 + c_2 x) = c_1 + c_2 x$

a) $\mathcal{L}(u_p) = f(x) = 1$

$u_p = \alpha_0$ won't work $u=1$ solves homog problem

$u_p = x(\alpha_1)$ won't work

$u_p = x^2 \alpha_0$

$\frac{du_p}{dx} = 2x\alpha_0 \quad \frac{d^2u_p}{dx^2} = 2\alpha_0$

$\mathcal{L}(u_p) = 2\alpha_0 = 1 \Rightarrow \alpha_0 = 1/2$

$u_p = \frac{1}{2}x^2$

and the gen soln is

$$\boxed{u(x) = c_1 + c_2 x + \frac{1}{2}x^2}$$

(19)

b) $f(x) = x$

$u_p = \alpha_1 x + \alpha_0$ won't work

$u_p = x(\alpha_1 x + \alpha_0)$ won't work

$u_p = x^2(\alpha_1 x + \alpha_0)$ will work

$$\frac{du_p}{dx} = 3\alpha_1 x^2 + 2\alpha_0 x$$

$$\frac{d^2u_p}{dx^2} = 6\alpha_1 x + 2\alpha_0$$

$$\frac{d^2u_p}{dx^2} = x$$

$$\overset{\shortparallel}{6\alpha_1 x + 2\alpha_0} \Rightarrow \alpha_1 = \frac{1}{6} \quad \alpha_0 = 0$$

$$u_p = x^2\left(\frac{1}{6}x\right)$$

and the general soln is

$$\boxed{u(x) = c_1 + c_2 x + \frac{1}{6}x^3}$$

c) $f(x) = x^2$

(over)

(20)

$u_p = \alpha_2 x^2 + \alpha_1 x + \alpha_0$ — Nope

$x(\alpha_2 x^2 + \alpha_1 x + \alpha_0)$ — Nope

$u_p = x^2(\alpha_2 x^2 + \alpha_1 x + \alpha_0)$ — will work.

$$\frac{du_p}{dx} = 4\alpha_2 x^3 + 3\alpha_1 x^2 + 2\alpha_0 x$$

$$\frac{d^2u_p}{dx^2} = 12\alpha_2 x^2 + 6\alpha_1 x + 2\alpha_0$$

$$\frac{d^2u_p}{dx^2} = x^2$$

$$\parallel$$

$$12\alpha_2 x^2 + 6\alpha_1 x + \alpha_0 \Rightarrow \alpha_2 = \frac{1}{12}$$

$$\alpha_1 = \alpha_0 = 0$$

$$u_p = x^2\left(\frac{1}{12}x^2\right)$$

and the general soln is

$$\boxed{u(x) = C_1 + C_2 x + \frac{1}{12}x^4}$$

(21)

$\mathcal{L}(u) = \frac{d^2u}{dx^2}$

d) $\mathcal{L}(u) = 6x^2 + 2x + 4$

(a) $4 \cdot \mathcal{L}(\frac{1}{2}x^2) \quad = \quad 4 \cdot 1$

(b) $2 \cdot \mathcal{L}(\frac{1}{6}x^3) \quad = 2 \cdot x$

$6 \cdot \mathcal{L}(\frac{1}{12}x^4) \quad = 6 \cdot x^2$

so $u_p = 6\frac{1}{12}x^4 + \frac{2}{6}x^3 + 4\frac{1}{2}x^2$

Solves $\mathcal{L}(u_p) = 6x^2 + 2x + 4$

so the general soln is

$$\boxed{u(x) = c_1 + c_2 x + \frac{1}{2}x^4 + \frac{1}{3}x^3 + 2x^2}$$

6 Use Duhamel

a) $\frac{d^2u}{dx^2} + u = \cos x = f(x)$

Solve homog problem $r^2 + 1 = 0$

$r = \pm i$ (over)

(22)

$u_k(x) = c_1 \cos x + c_2 \sin x$

BUT since IC's are at $x = z$

better to use

$u_k(x) = c_1 \cos(x-z) + c_2 \sin(x-z)$

$0 = u_k(z) = c_1 \cos(z-z) + c_2 \sin(z-z)$
$= c_1$

$\therefore c_1 = 0$

$1 = u_k'(z) = -c_1 \sin(z-z) + c_2 \cos(z-z)$
$= c_2$

$\therefore c_2 = 1$

So $u_k(x) = \sin(x-z)$

So, according to Duhamel

$$u(x) = \int_0^x \underbrace{\sin(x-z)}_{u_k} \underbrace{\cos(z)}_{f(z)} dz$$

DHq

(over)

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

$$\sin(a-b) = \sin a \cos b - \cos a \sin b$$

$$\Rightarrow \frac{\sin(a+b) + \sin(a-b)}{2} = \frac{2 \sin a \cos b}{2}$$

So $\sin a \cos b = \frac{1}{2}\left(\sin(a+b) + \sin(a-b)\right)$

Use this in DH a

$$u(x) = \frac{1}{2}\int_0^x \left(\sin(x-z+z) + \sin(x-z-z)\right) dz$$

$$= \frac{1}{2}\int_0^x \sin(x)\, dz + \frac{1}{2}\int_0^x \sin(x-2z)\, dz$$

We're integrating wrt z!

$$\frac{1}{2}\sin(x)\, z \Big|_0^x + \frac{1}{2}\,\frac{\cos(x-2z)}{+2}\Big|_0^x$$

$$= \frac{1}{2}\sin(x)\cdot x + \frac{1}{4}\left(\underbrace{\cos(-x) - \cos(x)}_{0}\right)$$

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So $\boxed{u(x) = \frac{1}{2} x \sin(x)}$

b) $\dfrac{d^2u}{dx^2} - u = e^{-x} = f(x)$

Solve for u_*
$$\begin{cases} \dfrac{d^2u_*}{dx^2} - u_* = 0 \\ u_*(2) = 0 \\ u_*'(2) = 1 \end{cases}$$

$r^2 - 1 = 0$
$r = \pm 1$

since ICs given at $x=2$ best to use

$u_*(x) = c_1 \cosh(x-2) + c_2 \sinh(x-2)$

$0 = u_*(2) = c_1$
$1 = u_*'(2) = c_2$

so $u_*(x) = \sinh(x-2)$ (over)

(25)

and Duhamel says

$$u(x) = \int_0^x \underset{\substack{\uparrow \\ u^*}}{\sinh(x-z)}\ \underset{\substack{\uparrow \\ f(z)}}{e^{-z}}\, dz$$

$$\sinh(x-z) = \frac{1}{2}\left(e^{x-z} - e^{z-x}\right) \quad \text{so}$$

$$u(x) = \frac{1}{2}\int_0^x \left(e^{x-z} - e^{z-x}\right) e^{-z}\, dz$$

$$= \frac{1}{2}\int_0^x e^{x-2z}\, dz - \frac{1}{2}\int_0^x e^{-x}\, dz$$

$$\frac{1}{2} e^x \frac{e^{-2z}}{-2}\Big|_0^x \qquad - \frac{1}{2} e^{-x} z \Big|_0^x$$

$$= -\frac{1}{4} e^x \left(e^{-2x} - 1\right) - \frac{1}{2} e^{-x} x$$

$$= \frac{1}{2}\left(\frac{e^x - e^{-x}}{2}\right) - \frac{1}{2} x e^{-x}$$

$$= \boxed{\frac{1}{2}\sinh(x) - 1/2\, x e^{-x}}$$

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c) $\dfrac{d^2y}{dx^2} = x^2 = f(x)$

Solve $\begin{cases} \dfrac{d^2u_*}{dx^2} = 0 \\ u_*(z) = 0 \\ u_*'(z) = 1 \end{cases}$ ~ $\begin{array}{l} r^2 = 0 \\ u_*(x) = c_1 + c_2(x-z) \\ 0 = u_*(z) = c_1 \\ 1 = u_*'(z) = c_2 \end{array}$

So $u_*(x) = x - z$

Duhamel says

$$u(x) = \int_0^x \underbrace{(x-z)}_{u_*} \underbrace{z^2}_{f(z)} \, dz$$

$$= \int_0^x x z^2 \, dz - \int_0^x z^3 \, dz$$

$$= x \left.\frac{z^3}{3}\right|_0^x - \left.\frac{z^4}{4}\right|_0^x$$

$$= \frac{x^4}{3} - \frac{x^4}{4} = \boxed{\frac{1}{12}x^4}$$

(27)

1) $\frac{d^2y}{dx^2} - \frac{dy}{dx} = e^x = f(x)$

Get u_* by solving

$$\frac{d^2u_*}{dx^2} - \frac{du_*}{dx} = 0 \qquad r^2 - r = 0$$

$$u_*(z) = 0 \quad \Rightarrow \quad r = \frac{1 \pm \sqrt{1-4\cdot 0}}{2}$$

$$u_*'(z) = 1 \qquad = \frac{1}{2} \pm \frac{1}{2}$$

$$u_*(x) = e^{x/2}(c_1 \cosh(x/2) + c_2 \sinh(x/2))$$

But I highly suggest use this instead

$$u_*(x) = e^{\frac{x-z}{2}}\left(c_1 \cosh\left(\frac{x-z}{2}\right) + c_2 \sinh\left(\frac{x-z}{2}\right)\right)$$

$$0 = u_*(z) = e^0(c_1 \cdot 1 + c_2 \cdot 0) = c_1$$

$$1 = u_*'(z) = e^0\left(\frac{1}{2} c_2 \cosh(0)\right) + \frac{1}{2} e^0 (c_1)$$

$c_1 = 0$

$c_2 = 2$

$$u_*(x) = 2e^{\frac{x-z}{2}} \sinh\left(\frac{x-z}{2}\right)$$ (over)

(28)

So Duhamel says

$$u(x) = \int_0^x \underbrace{2e^{\frac{x-z}{2}} \sinh\left(\frac{x-z}{2}\right)}_{u_K} \underbrace{e^{z}}_{f(z)} dz$$

$$= \int_0^x e^{\frac{x-z}{2}} \left(e^{\left(\frac{x-z}{2}\right)} - e^{-\left(\frac{x-z}{2}\right)}\right) e^{z} dz$$

$$= \int_0^x \left(e^{x-z} - 1\right) e^{z} dz$$

$$= \int_0^x e^{x} dz - \int_0^x e^{z} dz$$

$$e^{x} z \Big|_0^x - e^{z} \Big|_0^x$$

$$= e^{x} x - e^{x} + 1$$

So $\boxed{u(x) = xe^{x} - e^{x} + 1}$

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7) $$\frac{d^2u}{dx^2} - 2\frac{du}{dx} + 2u = \sin(x) = f(x)$$

Get u_* by solving

$$\frac{d^2u^*}{dx^2} - 2\frac{du^*}{dx} + 2u^* = 0 \qquad u_*(z) = 0, \quad u_*'(z) = 1$$

$$r^2 - 2r + 2 = 0$$

$$r = \frac{2 \pm \sqrt{4 - 4 \cdot 2}}{2} = 1 \pm i$$

Translate $x \to x - z$ b.c.

So $u_*(x) = e^{x-z}(c_1 \cos(x-z) + c_2 \sin(x-z))$

$$0 = u_*(z) = e^0(c_1 \cdot 1 + c_2 \cdot 0) = c_1$$

$$1 = u_*'(z) = e^0(-\cancel{c_1} \cdot 0 + c_2 \cdot 1) + e^0(c_1 \cdot 1 + c_2 \cdot 0) = c_2$$

So $u_* = e^{(x-z)} \sin(x-z)$

Duhamel says

$$u(x) = \int_0^x \underbrace{e^{(x-z)} \sin(x-z)}_{u_*} \underbrace{\sin(z)}_{f(z)} \, dz$$

(2)

$\cos(a+b) = \cos a \cos b - \sin a \sin b$

$\cos(a-b) = \cos a \cos b + \sin a \sin b$

$\cos(a-b) - \cos(a+b) = 2 \sin a \sin b$

so use $\sin a \sin b = \frac{1}{2}(\cos(a-b) - \cos(a+b))$

$$u(x) = \frac{1}{2}\int_0^x e^{x-z}\left(\cos((x-z)-z) - \cos((x-z)+z)\right) dz$$

$$= \frac{1}{2}\int_0^x e^{(x-z)} \cos(x-2z)\, dz$$

$$- \frac{1}{2}\int_0^x e^{(x-z)} \cos(x)\, dz$$

$$= \mathrm{I} - \mathrm{II}$$

II is easy

$$\frac{1}{2}\int_0^x e^{(x-z)} \cos(x)\, dz$$

$$= \frac{1}{2} e^x \cos(x) \int_0^x e^{-z}\, dz$$

$$= \frac{1}{2} e^x \cos(x) \left.\frac{e^{-z}}{-1}\right|_0^x = \text{(over)}$$

$$= \frac{1}{2} e^x \cos x \left(1 - e^{-x}\right)$$

II $$= \frac{1}{2} \cos x \left(e^x - 1\right)$$

I is way harder

$$I = \frac{1}{2} e^x \int_0^x e^{-z} \cos(2z - x)\, dz$$

But

The I by Parts twice trick — remember?

$$S = \int \underbrace{e^{-z}}_{dv} \underbrace{\cos(2z - x)}_{u}\, dz$$

$$= -e^{-z} \cos(2z-x) - \int e^{-z}\, 2\sin(2z-x)\, dz$$

$$= -e^{-z} \cos(2z-x) - 2\int \underbrace{e^{-z}}_{dv} \underbrace{\sin(2z-x)}_{u}\, dz$$

$$= -e^{-z} \cos(2z-x) - 2\left[-e^{-z} \sin(2z-x) + \int e^{-z}\, 2\cos(2z-x)\, dz\right]$$

(over)

(32)

$$S = -e^{-z}\cos(2z-x) + 2e^{-z}\sin(2z-x)$$

$$-4\underbrace{\int e^{-z}\cos(2z-x)\,dz}_{S}$$

So

$$5S = e^{-z}\left(2\sin(2z-x) - \cos(2z-x)\right)$$

So

$$\int_0^x e^{-z}\cos(2z-x)\,dz$$

$$= \frac{1}{5}\left[e^{-z}\left(2\sin(2z-x) - \cos(2z-x)\right)\right|_0^x$$

$$= \frac{1}{5}\left[e^{-x}\left(2\sin(x) - \cos(x)\right) - \left(2\sin(-x) - \cos(-x)\right)\right]$$

$$= \frac{1}{5}\left[e^{-x}\left(2\sin x - \cos x\right) + 2\sin x + \cos x\right]$$

Ans()

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So $I = \frac{1}{2}e^x \times \frac{1}{5}\left[e^{-x}(2\sin x - \cos x) + 2\sin x + \cos x\right]$

$= \frac{1}{10}\left[2\sin x - \cos x + 2e^x \sin x + e^x \cos x\right]$

So.

$u(x) = I - II$

$= \frac{1}{10}\left[\cancel{2\sin x - \cos x} + 2e^x \sin x + e^x \cos x\right] - \frac{1}{2}\cos x\,(e^x \cancel{-1})$

$= \frac{1}{10}\left[2\sin x - \cos x\right] + \frac{1}{2}\cos x$

$+ \frac{1}{10}\left[2e^x \sin x + e^x \cos x\right] - \frac{1}{2}e^x \cos x$

$= \frac{1}{5}\sin x + \frac{4}{10}\cos x + \frac{1}{5}e^x \sin x - \frac{4}{10}e^x \cos x$

(34)

so after all that work, I got

$$\boxed{u(x) = \frac{1}{5}\sin x + \frac{2}{5}\cos x + \frac{1}{5}e^x \sin x - \frac{2}{5}e^x \cos x}$$

wow! I checked my work using the method of guessing (much much faster!) and this answer seems to be correct.

8a) $\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} = x = f(x)$

Solve for u_*

C-E

$$\frac{d^2u_*}{dx^2} + \frac{1}{x}\frac{du_*}{dx} = 0 \qquad u_*(z) = 0$$

$$u_*'(z) = 1$$

$r(r-1) + r = 0$

$r^2 = 0 \quad r = 0 \pm 0$

so $u_*(x) = C_1 + C_2 \log(x)$

$0 = u_*(z) = C_1 + C_2 \log(z)$

$1 = u_*'(z) = C_2 \cdot \frac{1}{z}$ (over)

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get $c_2 = z$ and $c_1 = -z\log(z)$

so

$$u_*(x) = -z\log(z) + z\log(x)$$
$$= z(\log(x) - \log(z))$$
$$= z(\log(x/z))$$

so Duhamel says

$$u(x) = \int_1^x \underbrace{z(\log(x/z))}_{u_*} \underbrace{z}_{f(z)}\, dz$$

Will use integration by parts

$$\int_1^x \underbrace{\log(x/z)}_{u} \underbrace{z^2 dz}_{dv} \qquad \log(x) - \log(z)$$

$$= \log(x/z)\frac{z^3}{3}\Big|_1^x + \int_1^x \frac{1}{z}\frac{z^3}{3}\, dz$$

$$= \log(1)\frac{x^3}{3} - \log(x) + \frac{1}{3}\frac{z^3}{3}\Big|_1^x$$

(over)

(36)

$$\boxed{u(x) = -\frac{1}{3}\log(x) + \frac{1}{9}(x^3-1)}$$

8b) $\frac{d^2u}{dx^2} - \frac{1}{x}\frac{du}{dx} = x = f(x)$

So $\frac{d^2u_*}{dx^2} - \frac{1}{x}\frac{du_*}{dx} = 0 \qquad u_*(z) = 0$

$u_*'(z) = 1$

$r(r-1) - r = 0$

$r^2 - 2r = 0$

$r(r-2) = 0 \qquad r = 0,\ r = 2$

So $u_*(x) = c_1 + c_2 x^2$

$0 = c_1 + c_2 z^2$

$1 = 2c_2 z$ $\Rightarrow$ $c_2 = \frac{1}{2z}$

$c_1 = -\frac{1}{2z}z^2$

$u_*(x) = \frac{1}{2}\left(\frac{x^2}{z} - z\right)$

u_*

$f(z)$

Duhamel

$u(x) = \int_1^x \frac{1}{2z}(x^2 - z^2)\, z \, dz$

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Now let's do the integral

$$= \int_1^x \frac{1}{2}(x^2 - z^2)\,dz$$

$$= \frac{1}{2}\int_1^x x^2\,dz - \frac{1}{2}\int_1^x z^2\,dz$$

$$= \frac{1}{2}x^2 z \Big|_1^x - \frac{1}{2}\frac{z^3}{3}\Big|_1^x$$

$$= \frac{1}{2}\left(x^2(x-1) - \frac{1}{3}(x^3-1)\right)$$

$$= \frac{1}{6}\left(3x^2(x-1) - (x^3-1)\right)$$

$$= \frac{1}{6}\left(2x^3 - 3x^2 + 1\right)$$

$$\boxed{u(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{6}}$$

9a) $\frac{d^2u}{dx^2} + u = \cos x$

$u(0) = 1 \quad u'(0) = 2$

From 6a $\quad u_p(x) = \int_0^x \sin(x-z)\cos(z)\,dz$

$= \frac{1}{2}x\sin(x)$ Solves (over)

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$$\boxed{\frac{d^2u_p}{dx^2}+u_p=\cos x \qquad u_p(0)=u_p'(0)=0}$$

So to get initial values $u(0)=1$

$u'(0)=2$

Solve homogeneous problem

$$\frac{d^2u_h}{dx^2}+u_h=0 \Rightarrow u_h(x)=C_1\cos x+C_2\sin x$$

$$1=u_h(0)=C_1$$

$$2=u_h'(0)=C_2$$

$u_h(x)=\cos x+2\sin x$

So the solution to IVP 9a is

$$\boxed{u(x)=\cos x+2\sin x+\frac{1}{2}x\sin(x)}$$

9b) $\frac{d^2u}{dx^2}-u=e^{-x}$

$u(0)=1 \quad u'(0)=2$

From 6b $u_p(x)=\int_0^x \sinh(x-z)e^{-z}dz$

$$=\frac{1}{2}\sinh(x)-\frac{1}{2}xe^{-x}$$

solves $\left(\frac{d^2u_p}{dx^2}-u_p=e^{-x} \qquad u_p(0)=u_p'(0)=0\right)$

So solve homog problem

$$\frac{d^2u_h}{dx^2} - u_h = 0 \quad \text{with } u_h(0)=1$$
$$u_h'(0)=2$$

$\Rightarrow u_h(x) = \cosh(x) + 2\sinh(x)$

So the solution to IVP 9b is

$$\boxed{u(x) = \cosh(x) + 2\sinh(x) + \frac{1}{2}\sinh(x) - \frac{1}{2}xe^{-x}}$$

10a) $\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} = x$

$u(1)=1, \; u'(1)=2$

From 8a $\quad u_p(x) = \int_1^x z\left(\log\left(\frac{x}{z}\right)\right) z\,dz$

$$= -\frac{1}{3}\log(x) + \frac{1}{9}(x^3-1)$$

solves $\frac{d^2u_p}{dx^2} + \frac{1}{x}\frac{du_p}{dx} = x$

with $\quad u_p(1) = u_p'(1) = 0$

So, solve homogeneous problem with

ICs $\quad u_h(1)=1 \quad u_h'(1)=2 \quad$ (over)

(40)

$r(r-1)+1=0$

$u_h(x) = c_1 + c_2 \log(x)$

$1 = u_h(1) = c_1$

$2 = u_h'(1) = c_2$

$u_h(x) = 1 + \log(x)$ u_h u_p

so soln to 10a is

$$\boxed{u(x) = 1+\log(x) - \frac{1}{3}\log(x) + \frac{1}{9}(x^3-1)}$$

10b) $\frac{d^2u}{dx^2} - \frac{1}{x}\frac{du}{dx} = x \quad u(1)=1, \quad u'(1)=2$

From 8b $\quad u_p(x) = \int_1^x \frac{1}{2z}(x^2-z^2)z\,dz$

$$= \frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{6}$$

Solves $\frac{d^2u_p}{dx^2} - \frac{1}{x}\frac{du_p}{dx} = x \qquad u_p(1) = u_p'(1) = 0$

So solve homog problem with ICs

$u_h(1)=1 \quad u_h'(1)=2$ to get

$u_h(x) = x^2$ u_h u_p

so $\boxed{u(x) = x^2 + \frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{6}}$