Variational Calculus and the Brachistochrone Problem

Johann Bernoulli posed the problem of the *brachistochrone* to the readers of Acta Eruditorum in June, 1696. The problem statement reads as follows. What curve joining points A and B in the plane does a frictionless particle follow, initially at rest and influenced only by gravity, so as to **minimizes the time of traversal**? See the top of [1] for an excellent animation depicting the Brachistochrone Problem as well as for related historical facts and other details.

First some preliminaries. Set up coordinates so the vertical y-axis points downwards and the horizontal x-axis points to the right. Let $\mathbf{e}_x$ and $\mathbf{e}_y$ denote the unit vectors pointing in the x and y directions respectively. Take the point A at the origin and suppose B is to its right and below. Let $(x, y(x))$ be a continuous path joining A to B. That is

$$\begin{aligned}&A = (0,0),\ B = (x_B, y_B) \text{ with } x_B > 0 \text{ and } y_B > 0,\\ &y(0) = 0,\ y(x_B) = y_B.\end{aligned}$$

Visualize the path as an immovable wire joining A to B and the particle as a threaded bead allowed to slip along the wire frictionlessly. According to Newton's second law, the particle (i.e. bead) with position $(x(t), y(t))$ satisfies

$$\mathbf{x} \equiv \begin{pmatrix} x \\ y \end{pmatrix}, \quad m\frac{d^2\mathbf{x}}{dt^2} = mg\,\mathbf{e}_y + \mathbf{f}_\perp,$$

where m is the mass of the bead, g is the gravitational acceleration constant, and $\mathbf{f}_\perp$ is the normal force exerted by the wire on the bead to balance its weight and centripetal force due to curving motion. Conservation of energy is found by dotting the equation of motion by the particle's velocity vector

$$\begin{aligned}&m\frac{d^2\mathbf{x}}{dt^2}\cdot\frac{d\mathbf{x}}{dt} = (mg\,\mathbf{e}_y + \mathbf{f}_\perp)\cdot\frac{d\mathbf{x}}{dt} = mg\,\mathbf{e}_y\cdot\frac{d\mathbf{x}}{dt}\\ &\implies \quad \frac{d}{dt}\left(\tfrac{1}{2}v^2\right) = \frac{d}{dt}(gy) \quad \implies \quad \tfrac{1}{2}v^2 = gy + const,\end{aligned}$$

where the particle's speed is $v = ||d\mathbf{x}/dt||$. Initially $v = 0$ when $y = 0$, so we get

$$v = \sqrt{2gy}.$$

The time it takes the bead moving with speed v to slide down an infinitesimal segment of wire with length ds is given by

$$dt = \frac{ds}{v} = \frac{\sqrt{1+y_x^2}\ dx}{\sqrt{2gy}}.$$

[1] https://wikipedia.org/wiki/Brachistochrone_curve

Let's define

$$H(y, y_x) \equiv \frac{\sqrt{1+y_x^2}}{\sqrt{2gy}}.$$

Therefore, for a given smooth path $(x, y(x))$ joining $(0,0)$ to (x_B, y_B), the total traversal time can be written as

$$\mathcal{T}(y) \equiv \int_0^{x_B} H(y, y_x)\, dx.$$

FYI: $\mathcal{T}(y)$ is called a functional. Its argument isn't a point but actually a function, $y(x)$, coming from a certain class of functions, and for a given y, $\mathcal{T}$ returns a real number. The solution of the Brachistochrone Problem is found by determining the function y^* which minimizes $\mathcal{T}(y)$. That is, our goal is to find the function $y^*(x)$ with $y^*(0) = 0$, $y^*(x_B) = y_B$ such that

$$\mathcal{T}(y^*) \le \mathcal{T}(y) \text{ over all functions } y(x) \text{ satisfying } y(0) = 0,\ y(x_B) = y_B\,.$$

We'll show below that y^* solves a certain second order ODE with boundary conditions $y^*(0) = 0$, $y^*(x_B) = y_B$, and finally we'll solve the resulting BVP.

A technique known as the *Calculus of Variations* will be employed to determine the aforementioned BVP. To this end, let $\delta > 0$ and consider a smooth function $\phi(x)$ which is required to satisfy $\phi(0) = \phi(x_R) = 0$ but arbitrary otherwise. Let $y = y^* + \delta\phi$ and observe that such functions will define allowable paths joining points A to B. Therefore, since y^* is a minimizer of $\mathcal{T}(y)$

$$0 \le \frac{1}{\delta}\left(\mathcal{T}(y^* + \delta\phi) - \mathcal{T}(y^*)\right) = \frac{1}{\delta}\left(\int_0^{\delta} \frac{d}{d\epsilon}\mathcal{T}(y^* + \epsilon\phi)\, d\epsilon\right),$$

where the rightmost identity is found by applying the Fundamental Theorem of Calculus from Calculus 1. Next, calculate

$$\begin{aligned}\frac{d}{d\epsilon}\mathcal{T}(y^* + \epsilon\phi) &= \frac{d}{d\epsilon}\int_0^{x_B} H(y^* + \epsilon\phi, y_x^* + \epsilon\phi_x)\, dx = \int_0^{x_B} \frac{d}{d\epsilon}\, H(y^* + \epsilon\phi, y_x^* + \epsilon\phi_x)\, dx \\ &= \int_0^{x_B} \Big(H_y(y^* + \epsilon\phi, y_x^* + \epsilon\phi_x)\,\phi + H_{y_x}(y^* + \epsilon\phi, y_x^* + \epsilon\phi_x)\,\phi_x\Big)\, dx,\end{aligned}$$

where in the last identity I used the chain rule from Calculus 3. Specifically, given a function of two variables, $H(y, y_x)$ above, the chain rule states

$$\frac{d}{d\epsilon}H(a,b) = H_y(a,b)\frac{da}{d\epsilon} + H_{y_x}(a,b)\frac{db}{d\epsilon},$$

where it's common to use the notation

$$H_y(y, y_x) = \frac{\partial}{\partial y}H(y, y_x) \quad \text{and} \quad H_{y_x}(y, y_x) = \frac{\partial}{\partial y_x}H(y, y_x).$$

Let $\delta \to 0$, and use the fact that $\frac{1}{\delta}\int_0^\delta f(\epsilon)\,d\epsilon \to f(0)$ for any continuous f, to find

$$0 \le \int_0^{x_B} \Big(H_y(y^*, y_x^*)\,\phi + H_{y_x}(y^*, y_x^*)\,\phi_x \Big)\,dx.$$

Replace ϕ with $-\phi$ in this inequality to see we have actually shown

$$0 = \int_0^{x_B} \Big(H_y(y^*, y_x^*)\,\phi + H_{y_x}(y^*, y_x^*)\,\phi_x \Big)\,dx.$$

Since $\phi(0) = \phi(x_B) = 0$, integration by parts applied to the second integral term gives

$$0 = \int_0^{x_B} \Big(H_y(y^*, y_x^*) - \frac{d}{dx} H_{y_x}(y^*, y_x^*) \Big)\,\phi\,dx,$$

and from this and the fact that ϕ is otherwise arbitrary implies pointwise for $0 < x < x_B$

(EL-2nd)
$$H_y(y^*, y_x^*) - \frac{d}{dx} H_{y_x}(y^*, y_x^*) = 0$$
subject to $y^*(0) = 0$ and $y^*(x_B) = y_B$.

This is the celebrated *Euler-Lagrange* second order differential equation central to the theory of Variational Calculus; see [2]. $H(y, y_x)$ is called the *Lagrangian* and characterizes the given extrema problem. Before continuing on to our particular brachistochrone application, I want to show you how the second order Euler-Lagrange ODE above (where H does not depend explicitly on x) can be integrated once to a first order ODE via what is known as the *Beltrami identity*; see [3]. The Beltrami identity follows from a very clever application of the Calculus 3 chain rule and the usual product rule. Consider $H(y, y_x)$ where y is a smooth function of x, and use the chain rule to see that

$$\frac{d}{dx} H = H_y \frac{dy}{dx} + H_{y_x} \frac{d^2 y}{dx^2},$$

and use the product rule to see that

$$\frac{d}{dx}\left(H_{y_x} \frac{dy}{dx} \right) = H_{y_x} \frac{d^2 y}{dx^2} + \frac{d}{dx} H_{y_x} \frac{dy}{dx}.$$

Subtract the second from the first to get

$$\frac{d}{dx}\left(H - H_{y_x} \frac{dy}{dx} \right) = \left(H_y - \frac{d}{dx} H_{y_x} \right) \frac{dy}{dx}.$$

Therefore, given that y^* is a smooth minimizer of $\mathcal{T}(y)$, (EL-2nd) and Beltrami above combine to show

$$\frac{d}{dx}\left(H(y^*, y_x^*) - H_{y_x}(y^*, y_x^*) \frac{dy^*}{dx} \right) = 0$$

(EL-1st)
$$\implies \quad H(y^*, y_x^*) - H_{y_x}(y^*, y_x^*) \frac{dy^*}{dx} = const$$
subject to $y^*(0) = 0$ and $y^*(x_B) = y_B$.

[2] https://wikipedia.org/wiki/Euler-Lagrange_equation
[3] https://wikipedia.org/wiki/Beltrami_identity

Let's finish up by solving for the particular path, $(x, y^*(x))$, that solves the Brachistochrone Problem. From now on, I'm going to drop the $*$ superscript from the minimizer, y^*, to help simplify notation. Recall that

$$H(y, y_x) = \frac{\sqrt{1+y_x^2}}{\sqrt{2gy}} \quad \Longrightarrow \quad \frac{\partial}{\partial y_x} H(y, y_x) = \frac{y_x/\sqrt{1+y_x^2}}{\sqrt{2gy}}.$$

Insert these into the first order Euler-Lagrange ODE in (EL-1st) to find

$$\frac{\sqrt{1+y_x^2}}{\sqrt{2gy}} - \frac{y_x^2/\sqrt{1+y_x^2}}{\sqrt{2gy}} = c_1 \quad \Longrightarrow \quad ((1+y_x^2) - y_x^2) = c_1\sqrt{1+y_x^2}\sqrt{2gy}.$$

Redefine constant c_1 so that $c_1\sqrt{2g} \to c_1$ and do a little bit more algebra

$$y_x = \pm\sqrt{\frac{1/c_1^2 - y}{y}} = \pm\sqrt{\frac{c_1 - y}{y}},$$

where I've again redefined $1/c_1^2 \to c_1 > 0$. These separate

$$\pm\sqrt{\frac{y}{c_1 - y}}\, dy = dx.$$

Recall the y-axis points down and $y(0) = 0$. For small enough x we expect $y(x) > 0 \Rightarrow y_x \geq 0$. So we'll first take the $+$ sign in $\pm$ and march to the right from the left boundary condition $y(0) = 0$. Making the substitution $y = c_1 \sin^2(\theta) \Rightarrow dy = 2c_1 \sin(\theta)\cos(\theta)\, d\theta$ for θ in the range $0 \leq \theta \leq \pi/2$ get

$$\int_0^{x(\theta)} dx = 2c_1 \int_0^{\theta} \sqrt{\frac{c_1 \sin^2(\theta)}{c_1 - c_1\sin^2(\theta)}} \sin(\theta)\cos(\theta)\, d\theta$$

$$\Longrightarrow \qquad x(\theta) = 2c_1 \int_0^{\theta} \sin^2(\theta)\, d\theta = c_1(\theta - \tfrac{1}{2}\sin(2\theta)).$$

Note above that $x(\pi/2) = c_1\pi/2$ and $y(\pi/2) = c_1$ Also notice that y_x may change sign there, so we'll next take the $-$ sign in $\pm$, and using the same substitution as before, march right from $\pi/2$

$$\int_{x(\pi/2)}^{x(\theta)} dx = -2c_1 \int_{\pi/2}^{\theta} \sqrt{\frac{c_1 \sin^2(\theta)}{c_1 - c_1\sin^2(\theta)}} \sin(\theta)\cos(\theta)\, d\theta$$

$$\Longrightarrow \qquad x(\theta) - x(\pi/2) = 2c_1 \int_{\pi/2}^{\theta} \sin^2(\theta)\, d\theta = c_1(\theta - \pi/2 - \tfrac{1}{2}\sin(2\theta)).$$

Therefore, for all $0 \leq \theta \leq \pi$, we found a parameterized path

(C)
$$x(\theta) = c_1\left(\theta - \tfrac{1}{2}\sin(2\theta)\right),$$
$$y(\theta) = c_1\left(\tfrac{1}{2} - \tfrac{1}{2}\cos(2\theta)\right) \quad (= c_1\sin^2(\theta)),$$

which on changing variables satisfies the ODE in (EL-1st). It also satisfies the left boundary condition $y(0) = 0$ when $\theta = 0$. FYI. (C) describes a path known as a *cycloid*. It is a curve traced by a point on a circle as it rolls along a straight line without slipping; [4]. We still need to address how we'll get the right boundary condition. That is, how do you find $c_1 > 0$ and $0 < \theta_B < \pi$ such that $x(\theta_B) = x_B\,,\ y(\theta_B) = y_B$? See Exercise 1 below.

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1. Let $r(\theta) = x(\theta)/y(\theta)$ where $x(\theta)$ and $y(\theta)$ are given in (C). Also recall the boundary condition for $y(x)$: $y(x_B) = y_B$ where $x_B > 0$ and $y_B > 0$.
   (a) Show that $\lim_{\theta \to 0^+} r(\theta) = 0$.
   (b) Show that $\lim_{\theta \to \pi^-} r(\theta) = \infty$.
   (c) Conclude there is a $0 < \theta_B < \pi$ such that $r(\theta_B) = x_B/y_B$.
   (d) Determine the constant c_1 so that $x(\theta_B) = x_B$ and $y(\theta_B) = y_B$.

2. Let (x_A, y_A) and (x_B, y_B) with $x_A < x_B$ be two points in the plane. Also let $(x, y(x))$ denote a smooth path joining these. Use the Calculus of Variations to show the path with minimum arclength is in fact a straight line. Hint: Consider $\mathcal{A}(y) \equiv \int_{x_A}^{x_B} \sqrt{1 + y_x^2}\, dx$.

3. Find the minimum of $\mathcal{E}(y) \equiv \int_0^1 y_x^2\, dx$ over all functions $y(x)$ which satisfy $y(0) = 0$, $y(1) = 1$. Answer: $y^*(x) = x \ \Rightarrow\ \mathcal{E}(y^*) = 1$.

Perhaps a few closing questions and remarks may be of some value to the reader.

We found that the Euler-Lagrange equations are satisfied by a functional minimizer. But does that mean a solution to Euler-Lagrange is a minimizer? No way. Just like in calculus, when $\mathbf{x}_*$ minimizes a smooth scalar function $f(\mathbf{x})$ then $\nabla f(\mathbf{x}_*) = \mathbf{0}$ but not the other way around. $\mathbf{x}_*$ could be at a local maximum or saddle or one of several local minima. Much greater in-depth analysis is required to prove an Euler-Lagrange solution yields a global minimizer.

Is the Euler-Lagrange BVP always uniquely solvable? Existence and uniqueness for an IVP is almost always true and pretty easy to establish. Not so for BVPs. They are much harder to analyze. For example, consider

$$\frac{d^2u}{dx^2} + u = 0 \ \text{ with } u(0) = 0 \text{ and } u(\pi) = 0\,.$$

[4] `https://wikipedia.org/wiki/Cycloid`

Check that any multiple of $u = \sin(x)$ is a solution. No uniqeness here. Similarly, it's easy to show this example

$$\frac{d^2u}{dx^2} + u = 0 \ \text{ with } \ u(0) = 0 \ \text{ and } \ u(\pi) = 1$$

has no solution at all.

We've assumed the minimizer y^* to $\mathcal{T}(y)$ is a smooth function of x in order to deduce the second order Euler-Lagrange equation. But the functional $\mathcal{T}(y)$ is itself well defined even for functions which are for example continuous and piecewise linear. For such functions the Euler-Lagrange equation as a second order ODE doesn't make a whole lot of sense. There is a notion of *weak solutions* to Euler-Lagrange which can broaden the class of allowable minimizers. This topic is, however, way out of the scope of these notes.