2×2 Phase Plane Portraits and Linearization

A phase portrait depicts the trajectories of a system of ODEs in so-called phase space. We restrict our attention here to two dimensional *autonomous* first order systems

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix},$$

and in this case, phase space is simply the x-y plane. An autonomous ODE does not explicitly depend on the independent variable t, and this tells us when $(x(t), y(t))$ is a given solution to the ODE, then so is $(x(t+T), y(t+T))$ for any constant T. (Show this by changing the independent variable and use the chain rule.) Think of a trajectory as a path on which a particle with position $(x(t), y(t))$ traverses in time, both forward t and backward t, with its initial position at (x_0, y_0). Given the ODE is autonomous, a trajectory is a fixed curve in space. Also, the uniqueness criterion (i.e. the right hand side is Lipschitz) tells us distinct trajectory paths can never cross. Moreover, any point (x_0, y_0) lying on a given path can be used to identify that path.

Here's a very simple example. Let (x_0, y_0) be a point in the plane and consider the IVP

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 2y \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Its solution is clearly $(x(t), y(t)) = (e^t x_0, e^{2t} y_0)$. A trajectory containing (x_0, y_0) is given by the point set

$$p(x_0, y_0) = \{(x,y) = (e^t x_0, e^{2t} y_0) : -\infty < t < \infty\}$$

where here I've used the actual solution (as a t-parameterized curve) to describe it. Perhaps, however, this particular path can be better visualized by writing it instead as the graph of a function. The function definition will depend on where the point (x_0, y_0) is. So, for ease of presentation, let's take $x_0 > 0$.

$$x = x_0 e^t,\ y = y_0 e^{2t} \quad \Rightarrow \quad \frac{x}{x_0} = e^t \quad \Rightarrow \quad \left(\frac{x}{x_0}\right)^2 = e^{2t} \quad \Rightarrow \quad y = y_0\left(\frac{x}{x_0}\right)^2.$$

So when $x_0 > 0$, the path is given by the graph of $y = y_0(x/x_0)^2$ with $x > 0$. This defines the right half of a parabola which open upward when $y_0 > 0$ or downward when $y_0 < 0$.

It's often possible to determine a solution trajectory without first explicitly computing the ODE's solution. For example, in the previous example

$$\begin{aligned} \frac{dx}{dt} &= x \\ \frac{dy}{dt} &= 2y \end{aligned} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2y}{x} \quad \text{with } y(x_0) = y_0.$$

This separable scalar equation can be integrated to get

$$\log|y| - \log|y_0| = \log|x|^2 - \log|x_0|^2 \quad \Rightarrow \quad |y| = |y_0|\left(|x|/|x_0|\right)^2 .$$

However, I don't think doing it this way is always such a good idea, especially when it's possible to determine $(x(t), y(t))$ in closed form. For instance, a trajectory may not be globally defined by $y = y(x)$, or even by $x = x(y)$ for that matter. Try computing trajectories for

$$\begin{aligned} \frac{dx}{dt} &= x + y \\ \frac{dy}{dt} &= y - x \end{aligned} \qquad \text{by solving} \quad \frac{dy}{dx} = \frac{y-x}{x+y}$$

and you'll see what I mean.

Here's a second simple example. Let (x_0, y_0) be a point in the plane and consider the IVP

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Its solution is $(x(t), y(t)) = (e^t x_0, e^{-t} y_0)$ and defines the trajectory path

$$p(x_0, y_0) = \{(x, y) = (e^t x_0, e^{-t} y_0) : -\infty < t < \infty\}.$$

I'll again write this as the graph of a function to help visualize what these trajectories look like but, as before, only consider the case $x_0 > 0$.

$$x = x_0 e^t,\; y = y_0 e^{-t} \quad \Rightarrow \quad \frac{x}{x_0} = e^t \quad \Rightarrow \quad \left(\frac{x}{x_0}\right)^{-1} = e^{-t} \quad \Rightarrow \quad y = y_0\left(\frac{x_0}{x}\right).$$

So when $x_0 > 0$, the path is given by the graph of $y = y_0 x_0 / x$ with $x > 0$. This defines the right half of a hyperbola which lies above the x-axis when $y_0 > 0$ or below it when $y_0 < 0$.

You might want to observe that for both examples, the horizontal and vertical trajectories seem to emanate from $(0,0)$. The point $(x_c, y_c) = (0,0)$ is what is often called a *critical point*. More generally, a critical point (or alternately an *equilibrium point* or *stationary point*) is a zero of the right hand side of the ODE

$$\begin{pmatrix} f(x_c, y_c) \\ g(x_c, y_c) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For the two simple examples considered above, do the following. Draw a few selected trajectory paths taking the defining initial points (x_0, y_0) symmetrically distributed around the critical point $(0,0)$. Here're eight good choices

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} \pm 1 \\ \pm 1 \end{pmatrix}, \quad \begin{pmatrix} \pm 1 \\ \mp 1 \end{pmatrix}, \quad \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix}.$$

A phase portrait composed of trajectory curves alone hides some important information concerning how particles flow. Speed and direction of flow are not depicted. In the two phase portraits you just drew, add arrowheads to each of the eight trajectories to indicate the direction of flow in forward time. In the first example, notice that particles flow away from the critical point in all directions. Such a critical point is called an (unstable) *node*. In the second example, notice that particles flow away from the critical point along the x-axis but towards it along the y-axis. This critical point is called a *saddle point*.

Now let's consider the more general constant coefficient problem

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{1,1}\,x + a_{1,2}\,y \\ a_{2,1}\,x + a_{2,2}\,y \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

where the constant matrix A above is real. Notice again that $(x_c, y_c) = (0,0)$ is a critical point for this ODE, and it is the only one unless A has a zero eigenvalue (i.e. det(A)=0). For the time being, let's assume that A does not have a zero eigenvalue and moreover, for now, assume it is diagonalizable. The critical point $(0,0)$ can be classified into three basic categories. They are:

(1) *Node*, either stable or unstable.

(2) *Saddle*.

(3) *Spiral*, either stable or unstable.

Sometimes a fourth category (the *center*) is considered, but this really is a special case of a (neutral) spiral. Whether our critical point fits into category (1), (2) or (3) depends on the eigenvalues of the constant coefficient matrix A.

In our previous two homeworks we used the matrix e^{At} to solve constant coefficient IVPs, $\mathbf{x}(t) = e^{At}\,\mathbf{x}_0$. This solution technique however hides some fairly important geometrical information. Here on the other hand, we'll look for the solution by writing it in a different but relevant basis,

$$\mathbf{x}(t) = \alpha(t)\,\mathbf{b}_1 + \beta(t)\,\mathbf{b}_2.$$

Given an appropriate basis, trajectories in the α-β plane will be easily found. But the question is, how do we visualize such α-β coordinate trajectories when mapped as above to the x-y plane? It's actually pretty easy, and I'll explain how in the next paragraph.

A point in the α-β plane is written as

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \alpha\,\mathbf{e}_1 + \beta\,\mathbf{e}_2,$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ denote the standard unit basis vectors for $\mathbb{R}^2$. There are three fundamental linear transformation matrices to consider, scale S, horizontal shear H and rotation R,

$$S(s_1, s_2) \equiv \begin{pmatrix} |s_1| & 0 \\ 0 & |s_2| \end{pmatrix}, \quad H(\theta_h) \equiv \begin{pmatrix} 1 & \cos(\theta_h) \\ 0 & \sin(\theta_h) \end{pmatrix}, \quad R(\theta_r) \equiv \begin{pmatrix} \cos(\theta_r) & -\sin(\theta_r) \\ \sin(\theta_r) & \cos(\theta_r) \end{pmatrix}.$$

Let's see what each of these does to the unit viewbox $\mathcal{B} \equiv [-1,1] \times [-1,1]$. $S(s_1, s_2)$ takes $\mathcal{B}$ to a rectangle with sides scaled by $|s_1|$ and $|s_2|$, i.e. $[-|s_1|, |s_1|] \times [-|s_2|, |s_2|]$. The horizontal shear $H(\theta_h)$ takes $\mathcal{B}$ to a parallelogram with the top and bottom sides still horizonal but the left and right sides sheared to angle of θ_h (measured counter-clockwise from the $\mathbf{e}_1$ axis). Be aware, when $\pi < \theta_h < 2\pi$ the horizontal shear transformation will flip points in the box $\mathcal{B}$ upside-down. The rotation transformation $R(\theta_r)$ solidly rotates $\mathcal{B}$ around the origin by an angle of θ_r (also measured counter-clockwise from the $\mathbf{e}_1$ axis).

Here's how to visualize curves in the x-y plane when they're given in terms of α-β coordinates. Measure lengths $s_1 = ||\mathbf{b}_1||$ and $s_2 = ||\mathbf{b}_2||$, the angle θ_h between vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ and the angle θ_r vector $\mathbf{b}_1$ makes with the x-axis. Then, the linear transformation T given by

$$\begin{aligned} T \equiv R(\theta_r)\, H(\theta_h)\, S(s_1, s_2) \quad &\Rightarrow \quad T\,\mathbf{e}_1 = \mathbf{b}_1, \quad \text{and} \quad T\,\mathbf{e}_2 = \mathbf{b}_2 \\ \Rightarrow \quad T\,(\alpha\,\mathbf{e}_1 + \beta\,\mathbf{e}_2) &= \alpha\,\mathbf{b}_1 + \beta\,\mathbf{b}_2 = \mathbf{x}. \end{aligned}$$

Suppose your curves are written in α-β coordinates and their points are restricted to the unit box $\mathcal{B}$. Stretch or shrink the sides of the box and curves contained to a rectangle as prescribed above. Horizontally shear the rectangle and curves within it so that the resulting parallelogram is congruent to the parallelogram with four vertices $-\mathbf{b}_1 \pm \mathbf{b}_2$ and $\mathbf{b}_1 \pm \mathbf{b}_2$. Rotate the parallelogram and curves within so that the rotated $\mathbf{e}_1$ aligns with basis vector $\mathbf{b}_1$. That's it. You've got your phase portrait in the x-y plane.

Since for now we're assuming the constant coefficient matrix A is diagonalizable, it has two eigenvalues with associated linearly independent eigenvectors, say

$$A\mathbf{r}_1 = \lambda_1 \mathbf{r}_1 \quad \text{and} \quad A\mathbf{r}_2 = \lambda_2 \mathbf{r}_2.$$

Now make the following linear change of dependent variables

$$\mathbf{x} \equiv \begin{pmatrix} x \\ y \end{pmatrix} \quad \leftrightarrow \quad \widetilde{\mathbf{x}} \equiv \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \quad \text{via} \;\; \mathbf{x}(t) = \tilde{x}(t)\,\mathbf{r}_1 + \tilde{y}(t)\,\mathbf{r}_2.$$

Plug into the constant coefficient ODE

$$\frac{d\mathbf{x}}{dt} = \frac{d\tilde{x}}{dt}\,\mathbf{r}_1 + \frac{d\tilde{y}}{dt}\,\mathbf{r}_2 = A\mathbf{x} = \lambda_1 \tilde{x}\,\mathbf{r}_1 + \lambda_2 \tilde{y}\,\mathbf{r}_2.$$

Because $\mathbf{r}_1$ and $\mathbf{r}_2$ are independent, this decouples the ODE into

$$\frac{d\tilde{x}}{dt} = \lambda_1\,\tilde{x} \;\; \text{and} \;\; \frac{d\tilde{y}}{dt} = \lambda_2\,\tilde{y} \quad \Rightarrow \quad \tilde{x}(t) = e^{\lambda_1 t}\tilde{x}_0 \;\; \text{and} \;\; \tilde{y}(t) = e^{\lambda_2 t}\tilde{y}_0,$$

where $\tilde{x}_0$ and $\tilde{y}_0$ are (possibly complex) constants.

When A has real eigenvalues/eigenvectors, the relevant change of basis for the constant coefficient ODE's solution, $\mathbf{x}(t)$, is simply given by A's eigenvectors, $\mathbf{b}_1 = \mathbf{r}_1$, $\mathbf{b}_2 = \mathbf{r}_2$. That is, $\mathbf{x}(t) = \tilde{x}(t)\,\mathbf{r}_1 + \tilde{y}(t)\,\mathbf{r}_2$ is a real valued description of the solution. The cases when A has complex eigenvalues are a bit more tricky and will be treated later.

Suppose first that A's eigenvalues are real. Let $\widetilde{p}(\tilde{x}_0, \tilde{y}_0)$ denote a trajectory path of the ODE in the $\tilde{x}$-$\tilde{y}$ coordinate plane as a function of an initial point $(\tilde{x}_0, \tilde{y}_0)$ also in the $\tilde{x}$-$\tilde{y}$ plane,

$$\widetilde{p}(\tilde{x}_0, \tilde{y}_0) = \{\, (\tilde{x}(t), \tilde{y}(t)) = (e^{\lambda_1 t}\tilde{x}_0, e^{\lambda_2 t}\tilde{y}_0) : \ -\infty < t < \infty \,\}.$$

In tilde coordinates, this is very similar to example 1 or 2 discussed above. When $\tilde{x}_0 = 0$, the point set is described by a vertical half line

$$\widetilde{p}(0, \tilde{y}_0) = \begin{cases} \{\, (0, \tilde{y}) : \tilde{y} > 0 \,\} & \text{if } \tilde{y}_0 > 0, \\ \{\, (0, \tilde{y}) : \tilde{y} < 0 \,\} & \text{if } \tilde{y}_0 < 0. \end{cases}$$

When $\tilde{x}_0 \neq 0$, the point set is described by the graph of a function $\widetilde{y}(\tilde{x})$ whose domain is the half line $\tilde{x} < 0$ when $\tilde{x}_0 < 0$ or $\tilde{x} > 0$ when $\tilde{x}_0 > 0$

$$\tilde{x} = \tilde{x}_0\, e^{\lambda_1 t},\ \tilde{y} = \tilde{y}_0\, e^{\lambda_2 t} \quad \Rightarrow \quad \frac{\tilde{x}}{\tilde{x}_0} = e^{\lambda_1 t} \quad \Rightarrow \quad \left(\frac{\tilde{x}}{\tilde{x}_0}\right)^{\frac{\lambda_2}{\lambda_1}} = e^{\lambda_2 t} \quad \Rightarrow \quad \widetilde{y}(\tilde{x}) \equiv \tilde{y}_0 \left(\frac{\tilde{x}}{\tilde{x}_0}\right)^{\frac{\lambda_2}{\lambda_1}}.$$

That is, for $\tilde{x}_0 \neq 0$

$$\widetilde{p}(\tilde{x}_0, \tilde{y}_0) = \begin{cases} \{\, (\tilde{x}, \widetilde{y}(\tilde{x})) : \tilde{x} < 0 \,\} & \text{if } \tilde{x}_0 < 0, \\ \{\, (\tilde{x}, \widetilde{y}(\tilde{x})) : \tilde{x} > 0 \,\} & \text{if } \tilde{x}_0 > 0, \end{cases} \quad \text{where} \quad \widetilde{y}(\tilde{x}) = \tilde{y}_0 \left(\frac{\tilde{x}}{\tilde{x}_0}\right)^{\frac{\lambda_2}{\lambda_1}}.$$

The question now is, what does the graph of $\widetilde{y}(\tilde{x})$ look like? We'll treat these in subcases.

- λ_1 and λ_2 both positive.

All trajectories flow <u>away</u> from the critical point $(0,0)$. Their shapes depend on the ratio λ_2/λ_1. When the ratio is less than one, $\widetilde{y}(\tilde{x})$ has the gross appearance of a parabola, but lying on its side aligned along the $\tilde{x}$-axis, and it opens to the left or right. When greater than one, $\widetilde{y}(\tilde{x})$ is aligned with the $\tilde{y}$-axis, and it opens down or up. The critical point here is called an **unstable node**.

- λ_1 and λ_2 both negative.

Trajectories look just like in the previous case, but here all trajectories flow <u>toward</u> the critical point. In this case the critical point is called a **stable node**.

- λ_1 and λ_2 have different signs.

Here the ratio λ_2/λ_1 is negative, so

$$\widetilde{y}(\tilde{x}) = \tilde{y}_0 \left(\frac{\tilde{x}_0}{\tilde{x}}\right)^{\left|\frac{\lambda_2}{\lambda_1}\right|}$$

has the gross appearance of a hyperbola. When $\lambda_1 < 0 < \lambda_2$, the trajectories along the $\tilde{x}$-axis flow towards the origin, and the trajectories along the $\tilde{y}$-axis flow away. When $\lambda_2 < 0 < \lambda_1$, the trajectories along the $\tilde{y}$-axis flow towards the origin, and the trajectories along the $\tilde{x}$-axis flow away. A critical point of this form is called a **saddle**.

Here are two more (less) simple examples.

Consider the constant coefficient system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 2y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Clearly the matrix has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$. Therefore, the critical point $(0,0)$ is an unstable node. In $\tilde{x}$-$\tilde{y}$ coordinates, see that for $\tilde{x}_0 \neq 0$ its trajectories are given by the graphs of $\tilde{y} = \tilde{y}_0(\tilde{x}/\tilde{x}_0)^2$, parabolas which open up and down in the $\tilde{x}$-$\tilde{y}$ plane. The flow direction along all trajectories is away from $(0,0)$. This example is therefore exactly like our first simple example but in a different basis. Continuing, find the eigenvectors are

$$\lambda_1 = 1 \quad \Rightarrow \quad \mathbf{r}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \lambda_2 = 2 \quad \Rightarrow \quad \mathbf{r}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Note that $||\mathbf{r}_1|| = 1$ and $||\mathbf{r}_2|| = \sqrt{2}$, the horizontal shear angle is $\theta_h = 45°$ and rotation angle is $\theta_r = 0$; (recall the discussion on page 4.) Therefore, to visualize trajectories in the x-y plane, scale and shear your graphs accordingly in tilde coordinates, and you're done.

Consider another constant coefficient system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-2y \\ -y \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

This matrix has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$. Therefore, the critical point $(0,0)$ is a saddle. In $\tilde{x}$-$\tilde{y}$ coordinates, see that for $\tilde{x}_0 \neq 0$ its trajectories are given by the graphs of $\tilde{y} = \tilde{y}_0(\tilde{x}_0/\tilde{x})$, hyperbolas, and along the horizontal axis, $\tilde{y} = 0$, the forward time flow is outward away from the critical point $(0,0)$, and along the vertical axis, $\tilde{x} = 0$, the flow is inward toward the critical point. For $(\tilde{x}_0, \tilde{y}_0)$ strictly within the first quadrant of the $\tilde{x}$-$\tilde{y}$ plane, the flow along the hyperbolic trajectory that contains $(\tilde{x}_0, \tilde{y}_0)$ goes downwards and then turns to the right, etc. This example is exactly the same as our second simple example, again given in the eigenvector basis

$$\lambda_1 = 1 \quad \Rightarrow \quad \mathbf{r}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \lambda_2 = -1 \quad \Rightarrow \quad \mathbf{r}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Now to visualize this in x-y coordinates, the scale $S(1, \sqrt{2})$ and horizontal shear $H(45°)$ will map trajectories rendered in the $\tilde{x}$-$\tilde{y}$ plane to the x-y plane. (Again see page 4.)

Only a few words now about the cases not yet discussed for diagonalizable constant coefficeint systems with real eigenvalues. Specifically, the cases when one eigenvalue is zero and the other is nonzero. Here the critical point is not a point at all, it's actually a critical line since in such cases there is an eigenvector $\mathbf{r} \neq \mathbf{0}$ such that

$$A\mathbf{r} = 0\,\mathbf{r} \quad \Rightarrow \quad A\mathbf{x}_c = \mathbf{0} \;\text{ when }\; \mathbf{x}_c = \alpha\mathbf{r} \;\text{ for any }\; \alpha \in \mathbb{R}.$$

Therefore, the *critical line* is a straight line going through the origin in the direction $\mathbf{r}$. I'll let you think about this one, but it's easy.

Explaining how to draw phase portraits for problems with complex eigenvalues is more difficult compared to the real cases already done. Recall we wrote the constant coefficient ODE's general solution in the form

$$\mathbf{x}(t) = e^{\lambda_1 t}\tilde{x}_0\,\mathbf{r}_1 + e^{\lambda_2 t}\tilde{y}_0\,\mathbf{r}_2,$$

and when the coefficient matrix A is real, complex eigenvalues appear in conjugate pairs

$$\lambda_1 = \alpha - i\beta,\; \lambda_2 = \alpha + i\beta, \quad \Rightarrow \quad \mathbf{r}_1 = \mathbf{a} - i\mathbf{b},\; \mathbf{r}_2 = \mathbf{a} + i\mathbf{b},$$

where α, β and $\mathbf{a}$, $\mathbf{b}$ are all real. Moreover, the fact that $\mathbf{r}_1$ and $\mathbf{r}_2$ are linearly independent in $\mathbb{C}^2$ implies $\mathbf{a}$ and $\mathbf{b}$ are independent in $\mathbb{R}^2$. Plug these into the general solution and turn the crank to find

$$\begin{aligned}
\mathbf{x}(t) &= e^{(\alpha - i\beta)t}(\mathbf{a} - i\mathbf{b})\,\tilde{x}_0 + e^{(\alpha+i\beta)t}(\mathbf{a} + i\mathbf{b})\,\tilde{y}_0 \\
&= e^{\alpha t}(\cos(\beta t) - i\sin(\beta t))(\mathbf{a} - i\mathbf{b})\,\tilde{x}_0 + e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))(\mathbf{a} + i\mathbf{b})\,\tilde{y}_0 \\
&= e^{\alpha t}(\cos(\beta t)\,\mathbf{a} - \sin(\beta t)\,\mathbf{b})\,(\tilde{x}_0 + \tilde{y}_0) + e^{\alpha t}(\sin(\beta t)\,\mathbf{a} + \cos(\beta t)\,\mathbf{b})\,i(\tilde{y}_0 - \tilde{x}_0) \\
&= e^{\alpha t}\,(\cos(\beta t)\,\mathbf{a} - \sin(\beta t)\,\mathbf{b})\,\check{x}_0 + e^{\alpha t}\,(\sin(\beta t)\,\mathbf{a} + \cos(\beta t)\,\mathbf{b})\,\check{y}_0,
\end{aligned}$$

where I've redefined constants in the last identity $(\tilde{x}_0 + \tilde{y}_0) \to \check{x}_0$, $i(\tilde{y}_0 - \tilde{x}_0) \to \check{y}_0$. We finally have a real valued solution provided the constants $\check{x}_0$ and $\check{y}_0$ are real. Continue to rearrange terms and you'll find

$$\begin{aligned}
\mathbf{x}(t) &= e^{\alpha t}\,(\cos(\beta t)\,\check{x}_0 + \sin(\beta t)\,\check{y}_0)\;\mathbf{a} + e^{\alpha t}\,(-\sin(\beta t)\,\check{x}_0 + \cos(\beta t)\,\check{y}_0)\;\mathbf{b} \\
&= \check{x}(t)\,\mathbf{a} + \check{y}(t)\,\mathbf{b},
\end{aligned}$$

where

$$\check{\mathbf{x}}(t) \equiv \begin{pmatrix} \check{x}(t) \\ \check{y}(t) \end{pmatrix} = e^{\alpha t}\begin{pmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{pmatrix}\begin{pmatrix} \check{x}_0 \\ \check{y}_0 \end{pmatrix}.$$

In the $\check{x}$-$\check{y}$ coordinate plane, a trajectory path containing the point $(\check{x}_0, \check{y}_0)$ is given by the point set

$$\check{p}(\check{x}_0, \check{y}_0) = \{\,(\check{x}(t), \check{y}(t)) : -\infty < t < \infty\,\},$$

where $\check{\mathbf{x}}(t)$ is given above. This is a very tractable problem, and here's why. Consider the parametrization

$$\begin{pmatrix} \check{x}(t) \\ \check{y}(t) \end{pmatrix} = \begin{pmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{pmatrix} \begin{pmatrix} \check{x}_0 \\ \check{y}_0 \end{pmatrix} \equiv R(-\beta t)\, \check{\mathbf{x}}_0.$$

(Recall the rotation transformation matrix $R(\theta)$ is given on page 4.) $R(-\beta t)$ above is a time dependent rotation matrix applied to a constant vector $\check{\mathbf{x}}_0$. The flow $(\check{x}(t), \check{y}(t))$ is simply the constant point spinning around the origin (clockwise in forward time when $\beta > 0$ and counter-clockwise when $\beta < 0$) along a circle with constant radius. Multiplying this by the scalar function $e^{\alpha t}$,

$$\check{\mathbf{x}}(t) = e^{\alpha t} R(-\beta t)\, \check{\mathbf{x}}_0,$$

causes the radius of circular motion to become time dependent. In forward time, when $\alpha > 0$ the radius of motion grows. When $\alpha < 0$ the radius shrinks. Therefore, trajectories of $\check{\mathbf{x}}(t)$ spiral away from the origin when $\alpha > 0$, and they spiral in when $\alpha < 0$. When $\alpha = 0$ the motion is purely circular.

- $\lambda_1 = \alpha - i\beta$ and $\lambda_2 = \alpha + i\beta$ with $\alpha > 0$.

All trajectories spiral around and away from the critical point $(0,0)$. In $\check{x}$-$\check{y}$ coordinates they spiral clockwise in forward time when $\beta > 0$ and counter-clockwise when $\beta < 0$. In the x-y phase plane the spiral direction will reverse when the basis $\{\mathbf{a}, \mathbf{b}\}$ has left-hand orientation. This critical point is called an **unstable spiral**.

- $\lambda_1 = \alpha - i\beta$ and $\lambda_2 = \alpha + i\beta$ with $\alpha < 0$.

All trajectories spiral around and toward the critical point $(0,0)$. As above, the spiraling direction is either clockwise or counter-clockwise depending on the sign of β and the orientation of the basis $\{\mathbf{a}, \mathbf{b})$. The critical point here is called a **stable spiral**.

Here's a spiral example.

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y \\ x + y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

This matrix has eigenvalues $\lambda_1 = 1 - i = \alpha - i\beta$ and $\lambda_2 = 1 + i = \alpha + i\beta \;\Rightarrow\; \alpha = 1$ and $\beta = 1$. Therefore, the critical point is an unstable spiral. Let's go a bit further.

$$\lambda_1 = 1 - i \quad \Rightarrow \quad \mathbf{r}_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \mathbf{a} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\; \mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and from our discussion above

$$\mathbf{x} = \check{x}(t)\, \mathbf{a} + \check{y}(t)\, \mathbf{b} \;\text{ where }\; \check{\mathbf{x}}(t) = e^{t} R(-t)\, \check{\mathbf{x}}_0.$$

So, in $\check{x}$-$\check{y}$ coordinates, trajectories are clockwise rotating circular-like unstable spirals. To map these to the x-y plane (again recall the discussion on page 4) first note that $||\mathbf{a}|| = 1$

and $||\mathbf{b}|| = 1$, which gives scale matrix $S(1,1) = I$. The horizontal shear angle from basis vector $\mathbf{a}$ to $\mathbf{b}$ is $-90°$, and since $H(-90°)$ takes $\check{y}$ to $-\check{y}$, this transformation flips clockwise rotating motion to counter-clockwise. The final rotation transformation does not change the appearance of circular-like trajectories in any important way. Therefore, in x-y space, trajectories are counter-clockwise rotating circular-like unstable spirals.

Let me say just a few words concerning the case when A is not diagonalizable. Here there is only one eigenvalue, λ, and it's real, and a one dimensional eigenspace spanned by $\mathbf{r}$. For 2×2's in this case, any $\mathbf{g} \notin \text{span}\{\mathbf{r}\}$ is a generator of a Jordan chain,

$$\mathbf{r} \equiv (A - \lambda I)\mathbf{g} \quad \Rightarrow \quad A\mathbf{r} = \lambda\mathbf{r} \;\text{ and }\; A\mathbf{g} = \lambda\mathbf{g} + \mathbf{r},$$

where $\{\mathbf{g}, \mathbf{r}\}$ is an independent set of generalized eigenvectors. Use this basis to write

$$\mathbf{x}(t) = \tilde{x}(t)\,\mathbf{g} + \tilde{y}(t)\,\mathbf{r}, \quad \frac{d\mathbf{x}}{dt} = A\mathbf{x} \quad \Rightarrow \quad \frac{d\tilde{x}}{dt}\mathbf{g} + \frac{d\tilde{y}}{dt}\mathbf{r} = \lambda\tilde{x}\,\mathbf{g} + (\tilde{x} + \lambda\tilde{y})\,\mathbf{r}.$$

By independence, we must have

$$\frac{d\tilde{x}}{dt} = \lambda\tilde{x} \;\text{ and }\; \frac{d\tilde{y}}{dt} = \lambda\tilde{y} + \tilde{x},$$

which can be solved

$$\tilde{x}(t) = e^{\lambda t}\tilde{x}_0 \;\text{ and }\; \tilde{y}(t) = e^{\lambda t}\tilde{y}_0 + te^{\lambda t}\tilde{x}_0.$$

Trajectories are unstable when $\lambda > 0$ and stable when $\lambda < 0$. A phase portrait in $\tilde{x}$-$\tilde{y}$ coordinates is found with a little bit of effort.

- $\lambda > 0$ and A is not diagonalizable.

These critical points are called *unstable degenerate nodes.*

- $\lambda < 0$ and A is not diagonalizable.

These critical points are called *stable degenerate nodes.*

For the following three ODEs,

(a) Classify the critical point $(0,0)$.

(b) Make a rough x-y phase portrait. Include arrowheads to indicate flow direction.

1. $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x - y \\ y \end{pmatrix}$ 2. $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - 3y \\ -2y \end{pmatrix}$

3. $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2x + 4y \\ -2x + 2y \end{pmatrix}$

Hints: (1) Unstable node, $\lambda = 2$, $\mathbf{r} = (0,1)$, $\lambda = 1$, $\mathbf{r} = (1,1)$. (2) Saddle, $\lambda = 1$, $\mathbf{r} = (0,1)$, $\lambda = -2$, $\mathbf{r} = (1,1)$. (3) Neutral spiral, $\lambda_\pm = \pm i$, $\mathbf{r}_\pm = (1,0) \pm i\,(1,1)$.

Suppose the nonlinear ODE

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$$

has a critical point (x_c, y_c). That is, a point in phase space where $f(x_c, y_c) = g(x_c, y_c) = 0$. Apply Taylor's Theorem there to write

$$\begin{aligned}\begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} &= \begin{pmatrix} f(x_c,y_c) \\ g(x_c,y_c) \end{pmatrix} + \begin{pmatrix} f_x(x_c,y_c) & f_y(x_c,y_c) \\ g_x(x_c,y_c) & g_y(x_c,y_c) \end{pmatrix}\begin{pmatrix} x - x_c \\ y - y_c \end{pmatrix} + \cdots \\ &\approx \begin{pmatrix} f_x(x_c,y_c) & f_y(x_c,y_c) \\ g_x(x_c,y_c) & g_y(x_c,y_c) \end{pmatrix}\begin{pmatrix} x - x_c \\ y - y_c \end{pmatrix} \equiv A(\mathbf{x} - \mathbf{x}_c),\end{aligned}$$

where in the 2×2 matrix f_x, f_y, etc. denote the partial derivatives $\partial f/\partial x$, $\partial f/\partial y$, etc. evaluated at (x_c, y_c). Therefore,

$$\frac{d}{dt}(\mathbf{x} - \mathbf{x}_c) = \frac{d\mathbf{x}}{dt} \approx A(\mathbf{x} - \mathbf{x}_c).$$

A solution $\mathbf{u} \approx \mathbf{x} - \mathbf{x}_c$ to the linearized equation

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u}$$

can tell us a lot about the actual solution of the nonlinear ODE, particularly in a small neighborhood around $\mathbf{x}_c$.

Consider the example nonlinear problem,

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 - y^2 \\ x^2 - 1 \end{pmatrix}.$$

It has four critical points, $(x_c, y_c) = (1, \pm 1)$, $(x_c, y_c) = (-1, \pm 1)$. The matrix for the linearization is

$$\begin{pmatrix} 2x_c & -2y_c \\ 2x_c & 0 \end{pmatrix}. \quad \Rightarrow \quad \lambda = x_c \pm \sqrt{x_c^2 - 4x_c y_c}.$$

The critical points $(1, -1)$ and $(-1, 1)$ are saddle points. The points $(1, 1)$ and $(-1, -1)$ are spirals, the first is unstable and the second stable.

Find and classify all critical points for the following nonlinear ODEs.

4. $\displaystyle \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 - y^2 \\ xy - x \end{pmatrix}$ 5. $\displaystyle \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10x - 5xy \\ 3y + xy - 3y^2 \end{pmatrix}$

Answers: 4. $(-1, 1)$ stable node, $(1, 1)$ unstable node, $(0, 0)$ neutral degenerate node. 5. $(0, 0)$ unstable node, $(0, 1)$ saddle, $(3, 2)$ stable spiral.
