# Fundamental Set of Solutions

Consider the linear and homogeneous first order system of ODEs

$$\frac{d\mathbf{u}}{dt} = A(t)\,\mathbf{u}, \quad \text{where} \quad \mathbf{u} \in \mathbb{R}^n,\ A(t) \in \mathbb{R}^{n\times n},$$

and throughout assume the entries of $A(t)$, i.e. $a_{i,j}(t)$, are everywhere continuous real valued functions of the independent variable $t$. Suppose somehow we are able to determine a set containing $n$ solution vectors,

$$S \equiv \{\mathbf{u}_1(t),\ \mathbf{u}_2(t), \ldots, \mathbf{u}_n(t)\}, \quad \text{where for each } j = 1, 2, \ldots, n \text{ we have } \frac{d\mathbf{u}_j}{dt} = A(t)\,\mathbf{u}_j.$$

Here's what will be shown below. If $S$ is a linearly independent set of vectors at a single $t = t_*$, it is in fact linearly independent at every $t \in \mathbb{R}$. When so, $S$ is called a *fundamental set* of solutions. On the other hand, if $S$ is linearly dependent at a single $t$, it is in fact linearly dependent at all $t$.

To see this is true, consider an $n \times n$ matrix $W(t)$ constructed so its $j$th column is $\mathbf{u}_j(t)$,

$$W(t) = \begin{pmatrix} \mathbf{u}_1(t) & \mathbf{u}_2(t) & \cdots & \mathbf{u}_n(t) \end{pmatrix}, \quad \text{and let} \quad w(t) = \det W(t).$$

Often, $W(t)$ is called the *Wronskian matrix*, and $w(t)$ is called the *Wronskian determinant*. Later I'll show $w(t)$ solves the scalar ODE

$$\frac{dw}{dt} = \mathrm{Tr}(A(t))\,w,$$

where $\mathrm{Tr}(A)$ denotes the *trace* of the matrix $A$; $\mathrm{Tr}(A) \equiv a_{1,1} + a_{2,2} + \cdots + a_{n,n}$. You know from week two of our class how to explicitly solve this scalar ODE

$$w(t) = \exp(h(t) - h(t_*))\,w(t_*), \quad \text{where } h(t) \text{ is the antiderivative of } \mathrm{Tr}(A(t)).$$

We've assumed all entries of $A(t)$ are everywhere continuous, which implies $h(t)$ is a continuously differentiable function at every $t$, which implies $\exp(h(t) - h(t_*)) > 0$ for every $t$. Therefore

$$w(t) = 0 \quad \Longleftrightarrow \quad w(t_*) = 0.$$

But this is exactly what we need in order to conclude the set $\{\mathbf{u}_1(t),\ \mathbf{u}_2(t), \ldots, \mathbf{u}_n(t)\}$ is linearly independent/dependent at time $t$ if and only if it is so at some other $t = t_*$.

Now, to show $w$ solves the scalar ODE $dw/dt = \mathrm{Tr}(A)\,w$, I'll write the Wronskian matrix in terms of row vectors $\mathbf{r}_1(t), \ldots, \mathbf{r}_n(t)$,

$$W(t) = \begin{pmatrix} \mathbf{r}_1(t) \\ \vdots \\ \mathbf{r}_n(t) \end{pmatrix}, \quad \text{where} \quad (\mathbf{r}_i)_j = (\mathbf{u}_j)_i.$$

There's a pretty formula for the derivative of the determinant which was probably discovered by Leibniz. It says

$$\frac{dw}{dt} = \det\begin{pmatrix} (d/dt)\ \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_n \end{pmatrix} + \det\begin{pmatrix} \mathbf{r}_1 \\ (d/dt)\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_n \end{pmatrix} + \cdots + \det\begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ (d/dt)\ \mathbf{r}_n \end{pmatrix}.$$

This formula is sometimes called the *determinant product rule* and is derived by forming the difference quotient $\frac{1}{\Delta t}(w(t+\Delta t) - w(t))$ and then exploiting the multilinearity of the determinant. Try to derive this as an exercise. Next, use $u_{i,j} = (\mathbf{u}_j)_i$ and rewrite the $n$ systems of differential equations (i.e. with $j = 1, 2, \ldots, n$) in a row-wise manner to see

$$\frac{d\mathbf{r}_i}{dt} = \sum_{k=1}^{n} a_{i,k}(t)\,\mathbf{r}_k \quad \text{where} \quad (\mathbf{r}_i)_j = u_{i,j}.$$

Plug these into the Leibniz product rule formula to get

$$\frac{dw}{dt} = \det\begin{pmatrix} \sum_{k=1}^{n} a_{1,k}\,\mathbf{r}_k \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_n \end{pmatrix} + \det\begin{pmatrix} \mathbf{r}_1 \\ \sum_{k=1}^{n} a_{2,k}\,\mathbf{r}_k \\ \vdots \\ \mathbf{r}_n \end{pmatrix} + \cdots + \det\begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \sum_{k=1}^{n} a_{n,k}\,\mathbf{r}_k \end{pmatrix}.$$

In the first term on the right hand side above, use row multilinearity of the determinant to conclude

$$\det\begin{pmatrix} \sum_{k=1}^{n} a_{1,k}\,\mathbf{r}_k \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_n \end{pmatrix} = \sum_{k=1}^{n} a_{1,k}\ \det\begin{pmatrix} \mathbf{r}_k \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_n \end{pmatrix} = a_{1,1}\ \det\begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_n \end{pmatrix} = a_{1,1}\,w,$$

where above I used the fact that

$$\det\begin{pmatrix} \mathbf{r}_k \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_n \end{pmatrix} = 0 \ \text{ for all } \ k \neq 1.$$

Similarly

$$\det\begin{pmatrix} \mathbf{r}_1 \\ \sum_{k=1}^{n} a_{2,k}\,\mathbf{r}_k \\ \vdots \\ \mathbf{r}_n \end{pmatrix} = \sum_{k=1}^{n} a_{2,k}\ \det\begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_k \\ \vdots \\ \mathbf{r}_n \end{pmatrix} = a_{2,2}\ \det\begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_n \end{pmatrix} = a_{2,2}\,w.$$

Continue until $i = n$ to get

$$\det \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \sum_{k=1}^{n} a_{n,k}\, \mathbf{r}_k \end{pmatrix} = a_{n,n}\, w.$$

Therefore, putting these altogether, we finally arrive at

$$\frac{dw}{dt} = (a_{1,1} + a_{2,2} + \cdots + a_{n,n})\, w = \mathrm{Tr}(A)\, w,$$

which is what I said I would show.