(1)

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3331 E1

1 Classify

a) $x\frac{dy}{dx} + e^y = 0$

Not linear.

$$\frac{x\,dy}{e^y} = -\frac{dx}{x} \quad \boxed{\text{separable}}$$

$$\frac{dy}{dx} = -\frac{e^y}{x} \neq f\left(\frac{y}{x}\right) \quad \text{Not homog.}$$

$$\frac{\partial x}{\partial x} \stackrel{?}{=} \frac{\partial}{\partial y} e^y \quad \text{NOT} \quad \text{NOT exact}$$

b) $y^2 \frac{dy}{dx} + xy + x^2 = 0$

Not linear

$$y^2 \frac{dy}{dx} = -(xy + x^2) \quad \text{NOT separable}$$

$$\frac{dy}{dx} = -\frac{xy + x^2}{y^2} = -\left[\frac{x}{y} + \left(\frac{x}{y}\right)^2\right] = f\left(\frac{y}{x}\right)$$

$\boxed{\text{is Homog}}$

$$\frac{\partial}{\partial x} y^2 \stackrel{?}{=} \frac{\partial}{\partial y}(xy + x^2)$$

$$0 \neq x \quad \text{NOT EXACT}$$

②

c) $\frac{du}{dx} + u + x^2 = 0$ ~~1st order~~ $\boxed{\text{Linear (inhomog)}}$

Linear

$\frac{du}{dx} = -(x^2 - u)$ NOT separable

$\neq f(\frac{u}{x})$ NOT homog

$\frac{\partial 1}{\partial x} \stackrel{?}{=} \frac{\partial (u+x^2)}{\partial u}$ NOT exact

$0 \neq 1$

d) $(u + x + 1)\frac{du}{dx} + u + 2 = 0$

NOT Linear

$(u+x+1)\frac{du}{dx} = -(u+2)$ NOT separable

$\frac{du}{dx} = -\frac{u+2}{u+x+1} \neq f(\frac{u}{x})$ NOT homog

$\frac{\partial}{\partial x}(u+x+1) \stackrel{?}{=} \frac{\partial}{\partial u}(u+2)$

$1 = 1$ $\boxed{\text{IS EXACT}}$

2 Find explicit soln

a) $\frac{dy}{dx} = x^2(y^2+1)$

clearly separable

$$\int \frac{dy}{y^2+1} = \int x^2\,dx$$

$$\text{arctan}(y) = \frac{x^3}{3} + C$$

Solve for ~~y~~

$$\boxed{y = \tan\left(\frac{x^3}{3} + C\right)}$$

b) $xy\frac{dy}{dx} = y^2 + x^2$

$$\frac{dy}{dx} = \frac{y^2+x^2}{xy} = \left(\frac{y}{x}\right) + \left(\frac{x}{y}\right) \quad \text{homog}$$

$$\frac{y}{x} = v$$

$$\frac{d}{dx}(xv) = v + \frac{1}{v}, \quad x\frac{dv}{dx} + \cancel{v} = \cancel{v} + \frac{1}{v}$$

$$\int v\,dv = \int \frac{dx}{x} \Rightarrow \frac{v^2}{2} = \log(x) + C$$

-over-

$$v = \pm\sqrt{2\log x + c}$$

$$\boxed{u = \pm x\sqrt{2\log x + c}}$$

3 Find gen Soln

a) $\frac{du}{dx} = -u^2$ separable

$$\int \frac{du}{u^2} = -\int dx$$

$$\boxed{-\frac{1}{u} = -x + C} \quad \text{or} \quad u = \frac{1}{x+C}$$

b) $(u+x)\frac{du}{dx} + u + x^2 = 0$

$$\frac{\partial}{\partial x}(u+x) = \frac{\partial}{\partial u}(u+x^2) \qquad \text{EXACT.}$$

$$\frac{\partial \Lambda}{\partial u} = u + x, \quad \Lambda = \frac{u^2}{2} + xu + h(x)$$

$$\frac{\partial \Lambda}{\partial x} = 0 + u + h'(x) = u + x^2 \quad h'(x) = x^2$$

$$h(x) = \frac{x^3}{3}$$

$$\boxed{\frac{u^2}{2} + xu + \frac{x^3}{3} = C}$$

⑤

4 $$\frac{dz}{dt} = -k\sqrt{z}$$

$z(0) = 4$ & $z(1) = 1$

a) Use this to solve for k.

$$\int_{z=4}^{z=1} \frac{dz}{\sqrt{z}} = -\int_{t=0}^{t=1} k\,dt$$

$$= 2\sqrt{z}\Big|_4^1 = -k$$

$$2(1-2) = -2 \Rightarrow \boxed{k=2}$$

b) When is $z(t^*) = 0$?

$$\int_{z=4}^{z(t^*)=0} \frac{dz}{\sqrt{z}} = -2\int_{t=0}^{t=t^*} dt = -2t^*$$

$$\overset{\shortparallel}{2}(-2) \Rightarrow \boxed{t^* = 2}$$

(6)

5 Solve by given factorization

9) $\left(\frac{d}{dx} - I\right)\underbrace{\left(\frac{dy}{dx} - y\right)}_{v} = 0$

$$\frac{dv}{dx} - v = 0$$

$$e^{x}\frac{d}{dx}(e^{-x}v) = 0 \Rightarrow v = c_1 e^{x}$$

To get y

$$\underline{\frac{dy}{dx} - y = v = c_1 e^{x}}$$

$$\int \cancel{e^{x}}\frac{d}{dx}(e^{-x}y) = \int c_1 \cancel{e^{x}} = c_1 x$$

$$e^{-x}y = c_2 + c_1 x$$

$$\boxed{y = c_2 e^{x} + c_1 x e^{x}}$$

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b) $\left(\frac{d}{dx} - I\right)\underbrace{\left(\frac{dy}{dx} + y\right)}_{v} = 1$

$$\frac{dv}{dx} - v = 1$$

$$e^{x}\frac{d}{dx}(e^{-x}v) = 1 \Rightarrow \int \frac{d}{dx}(e^{-x}v) = \int e^{-x}$$

$$\Rightarrow e^{-x}v = c_1 - e^{-x}$$

$$v = c_1 e^{x} - 1$$

To get y solve

$$\frac{dy}{dx} + y = v = c_1 e^{x} - 1$$

$$\boxed{e^{-x}\frac{d}{dx}(e^{x}y)} = c_1 e^{x} - 1$$

$$\int \frac{d}{dx}(e^{x}y) = \int c_1 e^{2x} - e^{x}$$

$$e^{x}y = \frac{c_1}{2} e^{2x} - e^{x} + c_2$$

$$\boxed{y = \hat{c}_1 e^{x} + c_2 e^{-x} - 1}$$