①

Snders
3338 Exam 1

1a. To show $P(\phi)=0$

Clearly $\phi = \phi \cup \phi$ & $\phi = \phi \cap \phi$

$\Rightarrow P(\phi) = P(\phi \cup \phi) = P(\phi) + P(\phi)$

$\therefore P(\phi) = 0$

b. To show $P(E^c) = 1 - P(E)$

Since $S = E \cup E^c$ and $E \cap E^c = \phi$

$\Rightarrow 1 = P(S) = P(E \cup E^c) = P(E) + P(E^c)$

$\therefore P(E) = 1 - P(E^c)$

c. To show if $E_1 \subseteq E_2$ Then $P(E_1) \leq P(E_2)$

Since $E_2 = E_1 \cup (E_2 \cap E_1^c)$ and $E_1 \cap (E_2 \cap E_1^c) = \phi$

$\Rightarrow P(E_2) = P(E_1 \cup (E_2 \cap E_1^c)) = P(E_1) + P(E_2 \cap E_1^c)$

But $P(E_2 \cap E_1^c) \geq 0$

$\therefore P(E_2) \geq P(E_1)$

d. To show $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$E_1 \cup E_2 = (E_1 \cap E_2^c) \cup (E_2 \cap E_1^c) \cup (E_1 \cap E_2)$

Also note $(E_1 \cap E_2^c) \cap (E_2 \cap E_1^c) = \phi$

$(E_1 \cap E_2^c) \cap (E_1 \cap E_2) = \phi$ — over —

(2)

and $(E_2 \cap E_1^c) \cap (E_1 \cap E_2) = \phi$

(i.e. these sets are pairwise disjoint)

$$\Rightarrow_{(*)} P(E_1 \cup E_2) = P(E_1 \cap E_2^c) + P(E_2 \cap E_1^c) + P(E_1 \cap E_2)$$

We also have that

$$E_1 = E_1 \cap (E_2 \cup E_2^c) = (E_1 \cap E_2) \cup (E_1 \cap E_2^c)$$

and since these are disjoint

$$\Rightarrow P(E_1) = P(E_1 \cap E_2) + P(E_1 \cap E_2^c)$$

Similarly

$$E_2 = E_2 \cap (E_1 \cup E_1^c) = (E_2 \cap E_1) \cup (E_2 \cap E_1^c)$$

$$\Rightarrow P(E_2) = P(E_2 \cap E_1) + P(E_2 \cap E_1^c)$$

Plug these into (*) to get

$$P(E_1 \cup E_2) = (P(E_1) - P(E_1 \cap E_2)) + (P(E_2) - P(E_1 \cap E_2)) + P(E_1 \cap E_2)$$

$$= P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

③

$S = \{(H,H), (H,T), (T,H), (T,T)\}$

2a $1 = P(S)$

$= P(\{(H,H), (H,T), (T,H)\} \cup \{(T,T)\})$

$= \frac{3}{4} + P((T,T)) \Rightarrow \boxed{P((T,T)) = 1 - 3/4 = 1/4}$

b. $P(\{(H,H), (T,T)\}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

c. $P(\text{Coin 1} = H) = P(\{(H,H), (H,T)\}) = 1/2 \quad (*)$

$P(\text{Coin 2} = H) = P\{(H,H), (T,H)\}) = 1/2 \quad (**)$

$P(\{(H,H)\}) = \frac{1}{8}$

Use $(*)$ $\frac{1}{2} = P((H,H)) + P(H,T) = \frac{1}{8} + P((H,T))$

$$\boxed{P((H,T)) = 3/8}$$

Use $(**)$ $\frac{1}{2} = P((H,H)) + P((T,H)) = \frac{1}{8} + P(T,H)$

$$\boxed{P(T,H) = 3/8}$$

Finally $1 = P((H,H)) + P(H,T) + P(T,H) + P((T,T))$

$= \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + P(T,T)$

$$\boxed{P(T,T) = \frac{1}{8}}$$

3. $S = \{(H,H), (H,T), (T,H), (T,T)\}$

$$P((H,H)) = P(H,T) = P(T,H) = P(T,T) = \frac{1}{4}$$

$$X(s) = \begin{cases} 1 & s = (H,H) \\ 0 & s = (H,T) \\ 0 & s = (T,H) \\ -1 & s = (T,T) \end{cases} \qquad Y(s) = \begin{cases} 2 & s = (H,H) \\ 1 & s = (H,T) \\ 0 & s = (T,H) \\ -1 & s = (T,T) \end{cases}$$

$$\{s : X(s) \le x\} = \begin{cases} \phi & x < -1 \\ \{(T,T)\} & -1 \le x < 0 \\ \{(T,T), (H,T), (T,H)\} & 0 \le x < 1 \\ S & 1 \le x \end{cases}$$

$$\{s : Y(s) \le x\} = \begin{cases} \phi & x < -1 \\ \{(T,T)\} & -1 \le x < 0 \\ \{(T,T), (T,H)\} & 0 \le x < 1 \\ \{(T,T), (T,H), (H,T)\} & 1 \le x < 2 \\ S & 2 \le x \end{cases}$$

(a) $$F_X(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{4} & -1 \le x < 0 \\ 3/4 & 0 \le x < 1 \\ 1 & 1 \le x \end{cases} = \frac{1}{4} H(x+1) + \frac{1}{2} H(x-0) + \frac{1}{4} H(x-1)$$

optional

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b) $$F_X(x) = \begin{cases} 0 & x < -1 \\ 1/4 & -1 \le x < 0 \\ 1/2 & 0 \le x < 1 \\ 3/4 & 1 \le x < 2 \\ 1 & 2 \le x \end{cases} = \boxed{\begin{aligned} &\tfrac{1}{4}H(x+1) \\ &+ \tfrac{1}{4}H(x-0) \\ &+ \tfrac{1}{4}H(x-1) \\ &+ \tfrac{1}{4}H(x-2) \end{aligned}}$$

optional

c) $$f_X(x) = \frac{1}{4}\delta(x+1) + \frac{1}{2}\delta(x-0) + \frac{1}{4}\delta(x-1)$$

$$\mu_X = \int_{-\infty}^{\infty} x\left(\frac{1}{4}\delta(x+1) + \frac{1}{2}\delta(x-0) + \frac{1}{4}\delta(x-1)\right)$$

$$= \frac{1}{4}\int_{-\infty}^{\infty} x\,\delta(x+1) + \frac{1}{2}\int_{-\infty}^{\infty} x\,\delta(x-0) + \frac{1}{4}\int_{-\infty}^{\infty} x\,\delta(x-1)$$

$$= -\frac{1}{4} + \frac{0}{2} + \frac{1}{4} = 0$$

d) $$f_Y(x) = \frac{1}{4}\delta(x+1) + \frac{1}{4}\delta(x-0) + \frac{1}{4}\delta(x-1) + \frac{1}{4}\delta(x-2)$$

$$\mu_Y = \frac{1}{4}\int_{-\infty}^{\infty} x\left(\delta(x+1) + \delta(x-0) + \delta(x-1) + \delta(x-2)\right)$$

$$= \frac{1}{4}\left(-1 + 0 + 1 + 2\right) = \frac{1}{2}$$

⑥

4 $X \sim U(0,1) \Rightarrow F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1 \\ 1 & 1 \le x \end{cases}$

$Y = 2X - 1$

a. $F_Y(x) = P(Y \le x) \equiv \left(\begin{array}{l} \text{shorthand for} \\ P(\{s \in S \mid Y(s) \le x\}) \end{array} \right)$

$= P(2X - 1 \le x) = P(X \le (x+1)/2)$

$= \begin{cases} 0 & \frac{x+1}{2} < 0 \\ \frac{x+1}{2} & 0 \le \frac{x+1}{2} < 1 \\ 1 & 1 \le \frac{x+1}{2} \end{cases} = \begin{cases} 0 & x < -1 \\ \frac{x+1}{2} & -1 \le x < 1 \\ 1 & 1 \le x \end{cases}$

(distrib for $U(-1,1)$.)

b. $f_Y(x) = \frac{d}{dx} F_Y(x) = \begin{cases} 0 & x < -1 \\ 1/2 & -1 < x < 1 \\ 0 & 1 < x \end{cases}$

c. $\mu_Y = \int_{-\infty}^{\infty} x\, f_Y(x)\, dx = \int_{-1}^{1} x \cdot \frac{1}{2}\, dx = 0$

d. $\sigma_Y^2 = \int_{-\infty}^{\infty} (x-0)^2 f_Y(x)\, dx = \int_{-1}^{1} x^2 \cdot \frac{1}{2}\, dx = \frac{1}{3}$

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5.
$$f_{X,Y}(x,y) = \begin{cases} x+y & 0<x<1 \text{ \& } 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

a.

$P(0 < X \leq 1/2 \text{ and } 0 < Y \leq 1/2)$

$$= \int_0^{1/2}\int_0^{1/2} f_{X,Y}(x,y)\,dy\,dx = \int_0^{1/2}\int_0^{1/2}(x+y)\,dy\,dx$$

$$= \int_0^{1/2}\left(xy + \frac{y^2}{2}\right)\Big|_0^{1/2} dx = \frac{1}{2}\int_0^{1/2}(x+1/4)\,dx = \frac{1}{8}$$

b. $$P(X \leq x) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{X,Y}(u,v)\,dv\,du$$

clearly if $x < 0$ $P(X \leq x) = 0$

if $0 \leq x < 1$

$$P(X \leq x) = \int_0^x\int_0^1 (u+v)\,dv\,du = \frac{x}{2}(x+1)$$

if $x \geq 1$

$$P(X \leq x) = \int_0^1\int_0^1 (u+v)\,dv\,du = 1$$

⑧

b. (cont.) $P(X \le x) = \begin{cases} 0 & x < 0 \\ \frac{x}{2}(x+1) & 0 \le x < 1 \\ 1 & 1 \le x \end{cases}$

c. $P(Y \le y) = \int_{-\infty}^{y} \int_{-\infty}^{\infty} f_{X,Y}(u,v)\, du\, dv$

(note the order of integration here)

Again, clearly $y < 0$ $P(Y < y) = 0$

if $0 \le y < 1$ $P(Y \le y) = \int_{-\infty}^{y} \int_{-\infty}^{\infty} (u+v)\, du\, dv$

$= \frac{y}{2}(y+1)$

if $1 \le y$ $P(Y \le y) = 1$

So $P(Y \le y) = \begin{cases} 0 & y < 0 \\ \frac{y}{2}(y+1) & 0 \le y < 1 \\ 1 & 1 \le y \end{cases}$

d. Is $f_{X,Y}(x,y) \stackrel{?}{=} f_X(x)\, f_Y(y) =$

(is this true for all (x,y) ??) $= \left(\begin{cases} 0 & x < 0 \\ x + 1/2 & 0 \le x < 1 \\ 0 & 1 \le x \end{cases} \right) \left(\begin{cases} 0 & y < 0 \\ y + \frac{1}{2} & 0 \le y < 1 \\ 0 & 1 \le y \end{cases} \right)$

⑨ Let's take $(x, y) \in (0,1) \times (0,1)$

$LHS = x + y$

$RHS = (x + 1/2)(y + 1/2)$

$LHS \neq RHS \Rightarrow \underline{X}$ & $\underline{Y}$ are NOT independent.

$\left(\begin{array}{l} \text{you could have also shown that} \\ \quad F_{\underline{X},\underline{Y}}(x, y) \neq F_{\underline{X}}(x)\, F_{\underline{Y}}(y) \end{array} \right)$