3338) (P1

Here're some hints to prove

(a) $\lim_{x \to \infty} F_X(x) = 1$ (b) $\lim_{x \to -\infty} F_X(x) = 0$

What I'll show is

$\lim_{n \to \infty} F_X(n) = 1$ $\lim_{n \to -\infty} F_X(n) = 0$

as $n \to \pm\infty$ through the integers. Use the fact that F_X is an increasing function to conclude the limit is valid for any sequence of reals $x \to \pm\infty$.

Let $n \in \mathbb{Z}$ and define

$$E_n = \{s \in S : X(s) \le n\} \; (\subseteq S)$$

and note that

$$F_X(n) \equiv P(E_n).$$

$A \setminus B = A \cap B^c$

Also observe that $E_{n+1} \setminus E_n = \{s \in S : n < X(s) \le n+1\}$

This may help see other things needed below.

Prove the following: For any $n \in \mathbb{Z}$

(A) $S = (E_n) \cup (E_{n+1} \setminus E_n) \cup (E_{n+2} \setminus E_{n+1}) \cup \cdots$

$$= (E_n) \cup \bigcup_{k \geq n} (E_{k+1} \setminus E_k)$$

(B) $E_n = (E_n \setminus E_{n-1}) \cup (E_{n-1} \setminus E_{n-2}) \cup \cdots$

$$= \bigcup_{k \leq n} (E_k \setminus E_{k-1})$$

(C) Show each "bracketed" set above is disjoint from every other bracketed set.

Next, I'll show the following

(Fact) $\left[\forall \varepsilon > 0, \exists N \ni \forall n \geq N \quad \sum_{k=n}^{\infty} P(E_{k+1} \setminus E_k) < \varepsilon \right.$

As discussed in class (recall Cauchy sequences)

Since $P(E_{k+1} \setminus E_k) \geq 0$ (series of pos terms) all I need to do is show

$$\sum_{k=0}^{\infty} P(E_{k+1} \setminus E_k) \text{ is } \underline{\underline{\text{finite}}}.$$

(See your Calc II text book.)

With this goal in mind, let $n=0$, use (A) & (C) to get

$$1 = P(S) = P\left((E_0) \cup \bigcup_{k \geq 0} (E_{k+1} \setminus E_k)\right)$$

$$= P(E_0) + \sum_{k=0}^{\infty} P(E_{k+1} \setminus E_k)$$

$$\Rightarrow \sum_{k=0}^{\infty} P(E_{k+1} \setminus E_k) \leq 1.$$

∴ I can now use (Fact) from previous page.

Now, let $\varepsilon > 0$ but be otherwise arbitrary and determine $N \ni \forall n \geq N$ $\displaystyle\sum_{k=n}^{\infty} P(E_{k+1} \setminus E_k) < \varepsilon$

(From discussion above)

For any $n \geq N$, use (A) & (C) to get

$$1 = P(S) = P(E_n) + \sum_{k=n}^{\infty} P(E_{k+1} \setminus E_k)$$

$$< P(E_n) + \varepsilon$$

$$\Rightarrow \quad 1 - P(E_n) < \varepsilon \quad \forall n \geq N,$$

Moreover, since $P(E) \in [0,1]$ we also have

$$\Rightarrow \quad 0 \leq 1 - P(E_n)$$

This implies $\lim_{n\to\infty} (1 - P(E_n)) = 0$

$$\Rightarrow \lim_{n\to\infty} F_X(n) = \lim_{n\to\infty} P(E_n) = 1.$$

To show (b) on the top of page 1, use (B) and (C) with a similar argument.