Math 3338, Fa09: Homework 2

Let $F_X(x)$ denote the CDF for a random variable X and suppose its PDF $f_X \equiv dF_X/dx$ exists for all x. The mean of X, denoted by μ_X, is then given by the formula

$$\mu_X = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

when this integral exists. The variance of X is denoted by σ_X^2 and is given by the formula

$$\sigma_X^2 = \int_{-\infty}^{\infty} (x-\mu_X)^2 f_X(x)\,dx$$

when the integral exists. X is said to be *uniformly distributed* $U(a,b)$ $(a<b)$ when its CDF is given by

$$X \sim U(a,b) \quad \Leftrightarrow \quad F_X(x) = \begin{cases} 0 & \text{if } x<a \\ (x-a)/(b-a) & \text{if } a \le x < b \\ 1 & \text{if } b \le x. \end{cases}$$

(You may neglect the fact that this function is not differentiable at $x=a$ and $x=b$.)

1. Suppose $X \sim U(a,b)$. (a) Determine X's PDF. (b) Compute its mean μ_X. (c) Compute its variance σ_X^2. Answers: $\mu_X = (a+b)/2$, $\sigma_X^2 = (b-a)^2/12$.

2. Suppose $X \sim U(0,1)$ and let the random variable Y be given by $Y = 3X+4$. (a) Conclude that $Y \sim U(4,7)$ by determining its CDF. (b) Compute Y's mean and variance.

3. Suppose $X \sim U(0,1)$ and let the random variable Y be given by $Y = X^2$. (a) The random variable Y is not uniformly distributed here. Nevertheless, determine its CDF. (b) Compute Y's PDF. (c) Compute μ_Y. (I got $1/3$.) (d) Compute σ_Y^2. (I got $4/45$.)

4. Suppose $X \sim U(0,1)$ and let the random variable Y be given by $Y = -\log(X)/\lambda$ where the parameter $\lambda > 0$. (a) Derive that

$$F_Y(x) = \begin{cases} 0 & \text{if } x<0 \\ 1-e^{-\lambda x} & \text{if } x \ge 0. \end{cases}$$

Hint: $F_Y(x) = P(-\log(X)/\lambda \le x) = 1 - P(X < e^{-\lambda x}) = 1 - F_X(e^{-\lambda x})$. (This Y is said to be an exponentially distributed random variable.) (b) Compute Y's PDF. (c) Compute that $\mu_Y = 1/\lambda$. (d) Compute that $\sigma_Y^2 = 1/\lambda^2$.

A random variable X is said to be *normally distributed* $N(\mu,\sigma^2)$ when its PDF is given by

$$X \sim N(\mu,\sigma^2) \quad \Leftrightarrow \quad f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-(x-\mu)^2/2\sigma^2}.$$

From this one can write its CDF as

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-(y-\mu)^2/2\sigma^2}\, dy.$$

In what follows, you may assume that $\int_{-\infty}^{\infty} e^{-y^2}\, dy = \sqrt{\pi}$. Further reading about the normal distribution can be found at [1].

5. Suppose $X \sim N(\mu, \sigma^2)$. (a) Show that $\mu_X = \mu$. (b) Show that $\sigma_X^2 = \sigma^2$.

6. Suppose $X \sim N(0,1)$ and let $Y = 2X + 3$. (a) Show that Y is normally distributed. (b) Determine μ_Y and σ_Y^2. Answer: $\mu_Y = 3$, $\sigma_Y^2 = 4$.

7. Recall the spinner wheel introduced in the last homework assignment. We're going to modify it by placing a magnet behind the wheel at three o'clock, and we then observe that the spinner stops on average at three o'clock half of the time. Specifically, for any measurable subset $E \subseteq [0, 2\pi)$ we have

$$P(E) = \begin{cases} \frac{c}{2\pi} m(E) + \frac{1}{2} & \text{if } \pi/2 \in E \\ \frac{c}{2\pi} m(E) & \text{if } \pi/2 \notin E. \end{cases}$$

(a) What value must c be? Hint: $[0, 2\pi) = [0, \pi/2) \cup \{\pi/2\} \cup (\pi/2, 2\pi)$. (b) Let a random variable $X : [0, 2\pi) \to \mathbb{R}$ be defined by $X(s) = s/\pi$. Determine the CDF $F_X(x)$ and the PDF $f_X(x)$. Answer:

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ x/4 & \text{if } 0 \leq x < 1/2 \\ x/4 + 1/2 & \text{if } 1/2 \leq x \leq 2 \\ 1 & \text{if } x > 2, \end{cases} \qquad f_X(x) = \tfrac{1}{4}\chi_{[0,2]}(x) + \tfrac{1}{2}\delta(x - \tfrac{1}{2}).$$

(c) Consider the following game. You wager one dollar, spin the wheel, and are returned $X(s)$ dollars from the house. Compute the mean μ_X and determine how much money you win or lose on average per spin. Answer: $\mu_X = 3/4$.

You might be interested to read further on the Dirac delta function. The wiki found at [2] looks pretty good to me.

8. Gamblers noticed how often the wheel from the previous exercise stopped at three o'clock, and so they threatened the house proprieter with some serious bodily harm. In order to hide what he is doing, the proprieter replaced the original magnet with a much weaker one. Now the wheel stops at three o'clock only 1 out of 10 times on average. Determine what $P(E)$ is and then answer questions (b) and (c) above.

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[1] `http://en.wikipedia.org/wiki/Normal_distribution`
[2] `http://en.wikipedia.org/wiki/Dirac_delta_function`

9. Repeat the previous exercise, this time assuming two magnets are placed behind the wheel. One behind three o'clock and the second behind six o'clock. One average the wheel stops at three o'clock 1 out of 10 times and at six o'clock 1 out of 20 times.

10. This is an optional exercise. It is intended to demonstrate that the Dirac delta function behaves like a PDF to certain random variables as their variances tend to zero. With this in mind, let $h : \mathbb{R} \to \mathbb{R}$ be a given bounded and continuous function. Throughout, assume $\epsilon > 0$ and that it will tend to zero. (a) Suppose $U_\epsilon \sim U(a, a+\epsilon)$ and let f_{U_ϵ} denote its PDF. (b) Also suppose $N_\epsilon \sim N(\mu, \epsilon^2)$ and that its PDF is denoted by f_{N_ϵ}. Show that

$$\lim_{\epsilon \downarrow 0} \int_{-\infty}^{\infty} h(x) f_{U_\epsilon}(x)\, dx = h(a), \qquad \lim_{\epsilon \downarrow 0} \int_{-\infty}^{\infty} h(x) f_{N_\epsilon}(x)\, dx = h(\mu).$$

That is as ϵ tends to zero, $f_{U_\epsilon}(x)$ behaves like $\delta(x-a)$ and $f_{N_\epsilon}(x)$ behaves like $\delta(x-\mu)$.