Math 3338, Fa09: Homework 3

Consider a bag that contains two red marbles and two green marbles. You reach into the bag twice to draw out a marble. The sample space is $\{(R,R),(R,G),(G,R),(G,G)\}$ where the first component indicates the color of the first marble drawn and the second the color of the second. Now consider two probability tables.

$P(R,R)=\frac{2}{4}\frac{2}{4}=\frac{1}{4}$	$P(R,G)=\frac{2}{4}\frac{2}{4}=\frac{1}{4}$	$P(R,R)=\frac{2}{4}\frac{1}{3}=\frac{1}{6}$	$P(R,G)=\frac{2}{4}\frac{2}{3}=\frac{1}{3}$
$P(G,R)=\frac{2}{4}\frac{2}{4}=\frac{1}{4}$	$P(G,G)=\frac{2}{4}\frac{2}{4}=\frac{1}{4}$	$P(G,R)=\frac{2}{4}\frac{2}{3}=\frac{1}{3}$	$P(G,G)=\frac{2}{4}\frac{1}{3}=\frac{1}{6}$

The table on the left assumes a drawn marble is immediately replaced back into the bag, whereas the one on the right assumes drawn marbles are not replaced. Let E_1 denote the event that the first marble drawn is red, i.e. $E_1=\{(R,R),(R,G)\}$, and let E_2 denote the event that the second marble drawn is red, $E_2=\{(R,R),(G,R)\}$. Observe that

$$\begin{array}{ll} \text{marbles replaced} & \text{marbles not replaced} \\ P(E_1)=\frac{1}{4}+\frac{1}{4}=\frac{1}{2} & P(E_1)=\frac{1}{6}+\frac{1}{3}=\frac{1}{2} \\ P(E_2)=\frac{1}{4}+\frac{1}{4}=\frac{1}{2} & P(E_2)=\frac{1}{6}+\frac{1}{3}=\frac{1}{2}. \end{array}$$

Clearly, $P(R,R)=P(E_1\cap E_2)$. For the case of replacement (left), $1/4=P(E_1\cap E_2)=P(E_1)P(E_2)$, so events E_1 and E_2 are independent. On the other hand, for the no replacement case (right), $1/6=P(E_1\cap E_2)\neq P(E_1)P(E_2)$, so here these two events are not independent.

1. Let $E_{1,c}$ denote the event that the first marble drawn has color $c\in\{R,G\}$ and $E_{2,c}$ the event that the second drawn has color c. (a) In the replacement case above, show E_{1,c_1} is independent of E_{2,c_2} for every c_1 and c_2 in $\{R,G\}$. (b) In the no replacement case, show E_{1,c_1} is not independent of E_{2,c_2}. (c) Can $E_{1,c}$ be independent of itself?

2. Do exercise 21 on page 81 from your text.

3. Let E denote the event that all children in a family have the same sex. Let F denote the event that at most one child in a family is a boy. (a) The sample space for families with two children is $S=\{(B,B),(B,G),(G,B),(G,G)\}$. Assume each outcome is equally likely, i.e. $P(B,B)=1/4$ etc. Show that here E and F are not independent. (b) The sample space for families with three children is $S=\{(B,B,B),\ldots,(G,G,G)\}$, and again assume each outcome is equally likely. Show here E and F are independent.

4. Do exercise 22 on page 81 from your text.

5. A coin is flipped, with possible outcomes from $S_1 = \{H, T\}$, jointly with the roll of a die, with outcomes from $S_2 = \{1, 2, 3, 4, 5, 6\}$. The sample space $S = S_1 \times S_2$ to the joint action can be identified with the following table.

	$s_2 = 1$	$s_2 = 2$	$s_2 = 3$	$s_2 = 4$	$s_2 = 5$	$s_2 = 6$
$s_1 = H$	$(H, 1)$	$(H, 2)$	$(H, 3)$	$(H, 4)$	$(H, 5)$	$(H, 6)$
$s_2 = T$	$(T, 1)$	$(T, 2)$	$(T, 3)$	$(T, 4)$	$(T, 5)$	$(T, 6)$

It is observed in the joint action that $P(s_1 = H) = P(s_1 = T) = 1/2$ and $P(s_2 = 1) = \cdots = P(s_2 = 6) = 1/6$. (a) Does this imply that the probability of each event in the table is $1/12$? If not give an example. (b) Suppose $P(H, 1) = \cdots = P(T, 6) = 1/12$. Let $X : S \to \mathbb{R}$ denote the random variable whose value is one when $s_1 = H$ and zero when $s_1 = T$. Let $Y : S \to \mathbb{R}$ denote the random variable whose value is the number showing on the die. Determine the joint CDF $F_{X,Y}(x, y) = P(X \leq x, Y \leq y)$. Answer:

$$F_{X,Y}(x, y) = \frac{1}{12} \begin{cases} 0 & \text{if } x < 0 \text{ or } y < 1 \\ 1 & \text{if } 0 \leq x < 1 \text{ and } 1 \leq y < 2 \\ 2 & \text{if } 0 \leq x < 1 \text{ and } 2 \leq y < 3 \\ 3 & \text{if } 0 \leq x < 1 \text{ and } 3 \leq y < 4 \\ 4 & \text{if } 0 \leq x < 1 \text{ and } 4 \leq y < 5 \\ 5 & \text{if } 0 \leq x < 1 \text{ and } 5 \leq y < 6 \\ 6 & \text{if } 0 \leq x < 1 \text{ and } 6 \leq y \\ 2 & \text{if } 1 \leq x \text{ and } 1 \leq y < 2 \\ 4 & \text{if } 1 \leq x \text{ and } 2 \leq y < 3 \\ 6 & \text{if } 1 \leq x \text{ and } 3 \leq y < 4 \\ 8 & \text{if } 1 \leq x \text{ and } 4 \leq y < 5 \\ 10 & \text{if } 1 \leq x \text{ and } 5 \leq y < 6 \\ 12 & \text{if } 1 \leq x \text{ and } 6 \leq y, \end{cases}$$

and observe that this can be written much more compactly as

$$F_{X,Y}(x, y) = \frac{1}{12} \left(\sum_{i=0}^{1} H(x - i) \right) \left(\sum_{j=1}^{6} H(y - j) \right)$$

where $H(x)$ is the *Heaviside step function* (see for example [1]) discussed earlier in class. (c) Are X and Y independent? (d) Determine the joint PDF.

6. The function

$$f_{X,Y}(x, y) = \begin{cases} 6x^2 y & \text{if } 0 < x < 1 \text{ and } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

is the joint PDF for random variables X, Y. (a) Use this to compute the joint CDF

[1] `http://en.wikipedia.org/wiki/Heaviside_step_function`

$F_{X,Y}(x,y)$. Answer:

$$F_{X,Y}(x,y) = \begin{cases} 0 & \text{if } x \le 0 \text{ or } y \le 0 \\ \min(x,1)^3 \min(y,1)^2 & \text{if } x > 0 \text{ and } y > 0. \end{cases}$$

(b) Compute $P(0 < X \le 3/4, 1/3 < Y \le 2)$. Answer: $3/8$. (c) Compute the marginal distributions $F_X(x) = P(X \le x)$ and $F_Y(y) = P(Y \le y)$. (d) Are the random variables X and Y independent? (e) Compute the marginal density functions $f_X(x)$ and $f_Y(y)$.

7. The function

$$f_{X,Y}(x,y) = \begin{cases} x + y & \text{if } 0 < x < 1 \text{ and } 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

is the joint PDF for random variables X, Y. (a) Use this to compute the joint CDF $F_{X,Y}(x,y)$. Answer:

$$F_{X,Y}(x,y) = \begin{cases} 0 & \text{if } x \le 0 \text{ or } y \le 0 \\ \frac{1}{2}\min(x,1)\min(y,1)(\min(x,1) + \min(y,1)) & \text{if } x > 0 \text{ and } y > 0. \end{cases}$$

(b) Compute $P(0 < X \le 1/2, 0 < Y \le 1/2)$. Answer: $1/8$. (c) Compute the marginal distributions $F_X(x) = P(X \le x)$ and $F_Y(y) = P(Y \le y)$. (d) Are the random variables X and Y independent? (e) Compute the marginal density functions $f_X(x)$ and $f_Y(y)$.

8. Do exercise 8 on page 128 from your text.

9. In exercise 2 from homework 2 you computed that when $X \sim U(0,1)$ and $Y = 3X + 4$ then Y's PDF is given by

$$f_Y(x) = \begin{cases} 1/3 & \text{if } 4 < x < 7 \\ 0 & \text{otherwise} \end{cases} \qquad \Rightarrow \qquad \mu_Y = \int_4^7 x\,(1/3)\,dx = \frac{11}{2}.$$

Can you explain why $\mu_Y = \int_0^1 (3x+4)dx$ is no accident? Hint: Change variables in the integral above.

10. In exercise 4 from homework 2 you computed that when $X \sim U(0,1)$ and $Y = -\log(X)/\lambda$ then Y's PDF is given by

$$f_Y(x) = \begin{cases} 0 & \text{if } x < 0 \\ \lambda e^{-\lambda x} & \text{if } x \ge 0 \end{cases} \qquad \Rightarrow \qquad \mu_Y = \int_0^\infty x\left(\lambda e^{-\lambda x}\right)dx = \frac{1}{\lambda}.$$

Can you explain why $\mu_Y = \int_0^1 \left(-\log(x)/\lambda\right)dx$ is no accident? Hint: Change variables.