Math 3338, Fa09: Homework 5

The *moment generating function* for a random variable X is denoted by $M_X(t)$ and is defined by $M_X(t) = E[e^{tX}]$. For example, if $X \sim N(\mu, \sigma^2)$, then

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-(x-\mu)^2/2\sigma^2}\, dx.$$

It's a fun exercise (I'll do this in class) to evaluate the integral above to obtain

$$X \sim N(\mu, \sigma^2) \quad \Rightarrow \quad M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}.$$

Observe that the name moment generating function comes from the fact that

$$\frac{d^n}{dt^n} M_X(0) = \int_{-\infty}^{\infty} \frac{d^n}{dt^n} e^{tx} \,|_{t=0}\, f_X(x)\, dx = \int_{-\infty}^{\infty} x^n f_X(x)\, dx = E[X^n].$$

That is, the n^{th} moment $E[X^n]$ can be found by computing the n^{th} derivative of $M_X(t)$ evaluated at $t = 0$. Again using $X \sim N(\mu, \sigma^2)$ as an example

$$\frac{d}{dt} M_X(t) = \frac{d}{dt} e^{\mu t + \frac{1}{2}\sigma^2 t^2} = e^{\mu t + \frac{1}{2}\sigma^2 t^2} (\mu + \sigma^2 t)$$

$$\Rightarrow \quad E[X] = \frac{d}{dt} M_X(0) = \mu.$$

Differentiating once more gives

$$\frac{d^2}{dt^2} M_X(t) = \frac{d}{dt} (e^{\mu t + \frac{1}{2}\sigma^2 t^2} (\mu + \sigma^2 t)) = e^{\mu t + \frac{1}{2}\sigma^2 t^2} (\sigma^2 + (\mu + \sigma^2 t)^2)$$

$$\Rightarrow \quad E[X^2] = \frac{d^2}{dt^2} M_X(0) = \sigma^2 + \mu^2.$$

1. Suppose $X \sim U(a, b)$. Show that $M_X(t) = \frac{1}{b-a}\left(\frac{e^{bt} - e^{at}}{t}\right)$.

2. Suppose $X \sim B(n, p)$. Show that $M_X(t) = ((1-p) + e^t p)^n$. Hint: Use the binomial theorem.

3. Suppose $X \sim Pois(\lambda)$. Show that $M_X(t) = \exp(\lambda(e^t - 1))$.

4. Suppose $X \sim L(\mu, b)$. Show that $M_X(t) = \frac{e^{\mu t}}{1 - b^2 t^2}$ but is only defined for t in the range $-1/b < t < 1/b$.

Using the moment generating function, compute the first and second moments of X when

5. $X \sim U(a, b)$
6. $X \sim B(n, p)$
7. $X \sim Pois(\lambda)$
8. $X \sim L(\mu, b)$

(You can check your answers by using means and variances computed on earlier homework exercises.)

9. Suppose two events, say A and B, are independent. That is, $P(A \cap B) = P(A)P(B)$. Deduce the following. (a) Given $P(B) > 0$ we must have $P(A|B) = P(A)$. (b) Given $P(A) > 0$ we must have $P(B|A) = P(B)$.

10. Suppose two events, say A and B, are independent. (a) Prove that A and B^c are independent. (b) Prove that A^c and B are independent. (c) Prove that A^c and B^c are independent.