Math 3338, Fa09: Homework 6

1.(a) Suppose the moment generating function for a random variable X is given by $M_X(t) = \frac{1}{4} + \frac{1}{8}e^t + \frac{3}{8}e^{2t} + \frac{1}{4}e^{4t}$. Determine the PDF and CDF for X. Hint: Consider $\int_{-\infty}^{\infty} e^{xt}\delta(x-a)\,dx$. (b) Suppose $M_X(t) = \frac{1}{4} + \frac{3}{4}\frac{1}{1-t}$ defined for $t < 1$. Determine the PDF and CDF for this X. Hint: Consider $\int_0^{\infty} e^{xt}e^{-x}\,dx$.

2. Suppose a random variable X has $f_X(x)$ as its PDF, and suppose two jointly distributed random variables X, Y have joint PDF $f_{X,Y}(x,y)$. (a) If $a > 0$ and b are constants, show $f_{aX+b}(x) = f_X((x-b)/a)/a$. (b) Show $f_{X+Y}(x) = \int_{-\infty}^{\infty} f_{X,Y}(x-y,y)\,dy$. (c) Use part (a) to derive the moment generating function relationship $M_{aX+b}(t) = e^{bt}M_X(at)$. (d) Use part (b) to derive that when X and Y are independent then $M_{X+Y}(t) = M_X(t)M_Y(t)$.

3. Read the description of the *Monty Hall Problem* which can be found at [1]. Three random variables can be identified. Let $X_{car} \in \{1,2,3\}$ denote the door which the car is behind. Let $X_{picked} \in \{1,2,3\}$ denote the door the contestant originally picks. Assume that $P(X_{car} = k) = P(X_{picked} = k) = 1/3$ for $k = 1,2,3$ and also X_{car} and X_{picked} are independent. Let $X_{open} \notin \{X_{car}, X_{picked}\}$ denote the door Monty opens after the contestant picks his/her door. Note that this says Monty does not open the door with the car or the door picked by the contestant. Also note that it will turn out that how Monty makes his choice when there is one is irrelevant. Now, the law of total probability tells us

$$\begin{aligned} P(winning) =& P(winning \,|\, X_{picked} = X_{car})\, P(X_{picked} = X_{car}) \\ &+ P(winning \,|\, X_{picked} \neq X_{car})\, P(X_{picked} \neq X_{car}). \end{aligned}$$

(a) Determine $P(X_{picked} = X_{car})$ and $P(X_{picked} \neq X_{car})$. (b) Assume the contestant's strategy is to hold on to his/her pick after Monty opens his door. Determine $P(winning \,|\, X_{picked} = X_{car})$, $P(winning \,|\, X_{picked} \neq X_{car})$ and conclude $P(winning) = 1/3$. (c) Assume the contestant's strategy is to switch his/her pick after Monty opens his door. Determine $P(winning \,|\, X_{picked} = X_{car})$, $P(winning \,|\, X_{picked} \neq X_{car})$ and conclude $P(winning) = 2/3$.

4. Repeat the Monty Hall exercise above, except this time suppose there are four doors, with independent random variables $X_{car} \in \{1,2,3,4\}$ and $X_{picked} \in \{1,2,3,4\}$, where $P(X_{car} = k) = P(X_{picked} = k) = 1/4$ for $k = 1,2,3,4$. When the contestant switches, assume the two (rational) choices are made with equal likelihood. Answer: When not switching, $P(winning) = 1/4$. When switching, $P(winning) = 3/8$.

[1] http://en.wikipedia.org/wiki/Monty_Hall_problem

5. A certain college baseball player had pro scouts watching him bat 10% of his plate appearances during a season. For the full season the player hit .400. He was not real sharp when there were no scouts watching and at those times his batting average slipped to .380. What was the player's batting average when there were scouts watching? Answer: .580.

6. Suppose two random variables X and Y have the following joint PDF

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2}(x+y)e^{-(x+y)} & \text{if } x \geq 0 \text{ and } y \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the marginal CDFs $F_X(x)$ and $F_Y(y)$. Answer:

$$F_X(x) = \begin{cases} 1 - \frac{1}{2}(x+2)e^{-x} & \text{if } x \geq 0 \\ 0 & \text{otherwise,} \end{cases} \qquad F_Y(y) = \begin{cases} 1 - \frac{1}{2}(y+2)e^{-y} & \text{if } y \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(b) Determine the PDF $f_{X+Y}(u)$ and CDF $F_{X+Y}(z)$. Answer:

$$f_{X+Y}(u) = \begin{cases} \frac{1}{2}u^2 e^{-u} & \text{if } u \geq 0 \\ 0 & \text{otherwise,} \end{cases} \qquad F_{X+Y}(z) = \begin{cases} 1 - \frac{1}{2}((z+1)^2+1)e^{-z} & \text{if } z \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

7. If $X \sim N(0,1)$ (normal with mean 0 and variance 1), show that $X^2 \sim \chi_1^2$ (chi-squared with mean 1.) Hint: For $x \geq 0$, $P(X^2 \leq x) = P(-\sqrt{x} \leq X \leq \sqrt{x}) = \int_{-\sqrt{x}}^{\sqrt{x}} f_X(u)\,du$. Then see "Variable limits form" in [2]. Also $\Gamma(1/2) = \sqrt{\pi}$.

8. This exercise relates the binomial distribution to two independent Poisson distributed random variables. Suppose X and Y are independent random variables with $X \sim Pois(\alpha)$ and $Y \sim Pois(\beta)$, $\alpha > 0$ and $\beta > 0$. Also suppose $n \in \{0, 1, 2, \ldots\}$. (a) Show that

$$P(X+Y=n) = e^{-(\alpha+\beta)} \sum_{k=0}^{n} \frac{\alpha^k \beta^{n-k}}{k!\,(n-k)!} = e^{-(\alpha+\beta)} \frac{(\alpha+\beta)^n}{n!}.$$

Hint: Write $P((X,Y) \in \mathcal{R}) = \iint_{\mathcal{R}} f_X(x) f_Y(y)\,dxdy$ with $\mathcal{R} = \{(x,y) : x+y=n\}$. You may use that

$$\iint_{\mathcal{R}} \delta(x-a)\delta(y-b)\,dxdy = \begin{cases} 1 & \text{if } (a,b) \in \mathcal{R} \\ 0 & \text{if } (a,b) \notin \mathcal{R}. \end{cases}$$

(b) Show that

$$P(X \leq x \cap X+Y=n) = e^{-(\alpha+\beta)} \sum_{k=0}^{n} \frac{\alpha^k \beta^{n-k}}{k!\,(n-k)!} H(x-k).$$

(c) Suppose $Z = X \,|\, (X+Y=n)$. Show that $Z \sim B(n,p)$ where $p = \alpha/(\alpha+\beta)$.

[2] http://en.wikipedia.org/wiki/Leibniz_integral_rule