

MATH 3363 FINAL EXAM. Sanders Fall '02

This exam has 10 problems and all 10 problems will be graded. You have a full three hours to complete it. Use my supplied paper only and return your solution sheets with the problems in order. Put your name, **last name first**, and **social security number** on each solution sheet you turn in. Good luck.

1. Solve the following boundary value problems.

$$\begin{array}{ll} \text{(a)} \quad \frac{d^2 y}{dx^2} - y = 0 & \text{(b)} \quad \frac{d^2 y}{dx^2} + \frac{dy}{dx} = 0 \\ y(0) = 1 \quad y'(1) = 0 & y(0) = 0 \quad y(1) = 1 \end{array}$$

2. Define the differential operator $\mathcal{L}(u) = \frac{d^2 u}{dx^2}$. Derive the eigenvalues and eigenfunctions for \mathcal{L} subject to the following boundary conditions.

$$\text{(a)} \quad u'(0) = 0 \quad u(1) = 0 \quad \text{(b)} \quad u(-\pi) = u(\pi) \quad u'(-\pi) = u'(\pi)$$

3a. Determine the 2π -periodic Fourier series for the function $f(x) = 1 + \sin^2(x)$.

3b. Determine the 2π -periodic Fourier series for the function $f(x) = x$.

3c. Use Parseval and the result of part (b) to derive $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

4. Consider the set of basis functions $\{\phi_{m,n}\}$ where $\phi_{m,n}(x, y) = \cos(m\pi x) \sin(n\pi y)$ with indices ranging from $m = 0, 1, 2, \dots$ and $n = 1, 2, \dots$

(a) Show $\int_0^1 \int_0^1 \phi_{m',n'}(x, y) \phi_{m,n}(x, y) dx dy = 0$ for any index pair $(m', n') \neq (m, n)$.

(b) Show $\int_0^1 \int_0^1 \phi_{m,n}(x, y) \phi_{m,n}(x, y) dx dy = \begin{cases} 1/2 & \text{if } m = 0 \\ 1/4 & \text{otherwise.} \end{cases}$

(c) Expand the function $f(x, y) \equiv 1$ in terms of this basis.

5. Solve the one dimensional heat equation:

$$\begin{array}{ll} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} & \text{with } \underline{\text{periodic}} \text{ boundary conditions} \quad \begin{array}{l} u(-\pi, t) = u(\pi, t) \\ u_x(-\pi, t) = u_x(\pi, t), \end{array} \\ \text{and initial condition} & u(x, 0) = 1 + \sin(2x) + \cos(4x). \end{array}$$

6. Solve the one dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad \text{with boundary conditions} \quad \begin{array}{l} u_x(0, t) = 0 \\ u_x(1, t) = 0, \end{array}$$

$$\text{and initial conditions} \quad \begin{array}{l} u(x, 0) = 1 + \cos(4\pi x) \\ u_t(x, 0) = 1. \end{array}$$

7. Solve the given two dimensional heat equation.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad \text{on the unit box } 0 \leq x \leq 1, 0 \leq y \leq 1,$$

$$\text{subject to boundary conditions } \begin{array}{ll} u_x(0, y, t) = 0 & u(x, 0, t) = 0 \\ u_x(1, y, t) = 0 & u(x, 1, t) = 0, \end{array}$$

$$\text{and initial condition } u(x, y, 0) = 1.$$

(Hint: Use the result from problem 4c.)

8. Solve Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ on the unit square $0 < x < 1, 0 < y < 1$ subject to the following boundary conditions.

$$\begin{array}{ll} \text{(a)} & \begin{array}{l} u(x, 0) = 0 \quad u(x, 1) = 0 \\ u(0, y) = 0 \quad u(1, y) = \sin(5\pi y) \end{array} \\ \text{(b)} & \begin{array}{l} u(x, 0) = 0 \quad u(x, 1) = \sin(\pi x) \\ u(0, y) = 0 \quad u(1, y) = 0 \end{array} \\ \text{(c)} & \begin{array}{l} u(x, 0) = 0 \quad u(x, 1) = \sin(\pi x) \\ u(0, y) = 0 \quad u(1, y) = \sin(5\pi y) \end{array} \end{array}$$

9. Find the solution $u = u(r, \theta)$ of Laplace's equation $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$ on the following domains satisfying the given boundary conditions.

$$\begin{array}{ll} \text{(a)} & \begin{array}{l} \text{The unit disk } r \leq 1 \\ \text{with } u(1, \theta) = 1 + \cos(\theta). \end{array} \\ \text{(a)} & \begin{array}{l} \text{The annulus } 1 \leq r \leq 2 \\ \text{with } u(1, \theta) = 1 + \sin(\theta), \\ \text{and } u(2, \theta) = 1. \end{array} \end{array}$$

10. Here's a free 20 points. Y'all take care!