

MATH 3363 FINAL EXAM. Sanders Fall 2003

This exam has 10 problems and all 10 problems will be graded. You have a full three hours to complete it. Use my supplied paper only and return your solution sheets with the problems in order. Put your name, **last name first please**, and **social security number** on each solution sheet you turn in. Good luck.

1. Solve the given eigenvalue problem to determine **all** real eigenvalues and associated eigenfunctions. (You may assume  $\lambda$  is real, but nothing more.)

$$\frac{d^2u}{dx^2} = \lambda u \quad \text{with} \quad u(-\pi) = u(\pi), \quad u_x(-\pi) = u_x(\pi)$$

2. Determine the  $\sin(n\pi x)$  Fourier series for the following functions.

$$(a) f(x) = \sin(\pi x) \cos(\pi x) \quad (b) f(x) = x$$

Hint: On part (a) use a simple trig identity.

3. Use the result you obtained in 2(b) above together with Parseval's identity to derive

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

4. Solve the one dimensional heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{with b.c. } u(0, t) = 0, \quad u_x(1, t) = 0$$

$$\text{and i.c. } u(x, 0) = 1 + \sin(3/2\pi x) + \sin(7/2\pi x).$$

(Be careful about the 1 in the initial data.)

5. Solve the one dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad \text{with b.c. } u_x(0, t) = 0, \quad u_x(1, t) = 0$$

$$\text{and i.c. } u(x, 0) = 1 + \cos(4\pi x), \quad u_t(x, 0) = 1.$$

6. Consider the inhomogeneous heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 2, \quad \text{with b.c. } u(0, t) = 0, \quad u(1, t) = 0,$$

$$\text{and i.c. } u(x, 0) = x^2 - x + \sin(2\pi x).$$

- (a) Determine the steady-state solution.

- (b) Solve for  $u(x, t)$ .

7. Solve the following first order equations on the domain  $-\infty < x < \infty$  and  $t \geq 0$  by the Fourier transform method.

$$(a) \frac{\partial u}{\partial t} + 2 \frac{\partial u}{\partial x} = 0 \text{ with } u(x, 0) = f(x) \quad (b) \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + 2u = 0 \text{ with } u(x, 0) = f(x)$$

8. List all eigenfunctions and corresponding eigenvalues to the Laplacian ( $\nabla^2 u \equiv u_{xx} + u_{yy}$ ) on the unit square  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$  with the given boundary conditions. (You must enumerate your eigenvalues–eigenfunctions correctly to receive full credit.)

$$(a) \begin{array}{l} u(0, y) = 0 \\ u(1, y) = 0 \\ u(x, 0) = 0 \\ u(x, 1) = 0 \end{array} \quad (b) \begin{array}{l} u_x(0, y) = 0 \\ u_x(1, y) = 0 \\ u_y(x, 0) = 0 \\ u_y(x, 1) = 0 \end{array} \quad (c) \begin{array}{l} u(0, y) = 0 \\ u_x(1, y) = 0 \\ u(x, 0) = 0 \\ u_y(x, 1) = 0 \end{array}$$

9. Solve Laplace's equation  $u_{xx} + u_{yy} = 0$  on the unit square  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$  subject to the following boundary conditions.

$$(a) \begin{array}{l} u(x, 0) = 0 \\ u(0, y) = 0 \end{array} \quad \begin{array}{l} u(x, 1) = 0 \\ u(1, y) = \sin(5\pi y) \end{array} \quad (b) \begin{array}{l} u(x, 0) = 0 \\ u(0, y) = 0 \end{array} \quad \begin{array}{l} u(x, 1) = \sin(\pi x) \\ u(1, y) = 0 \end{array}$$

10. Find the solution  $u = u(r, \theta)$  of Laplace's equation  $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$  on the following domains satisfying the given boundary conditions.

$$(a) \begin{array}{l} \text{The unit disk } r \leq 1 \\ \text{with } u(1, \theta) = 1 + \sin(\theta). \end{array} \quad (b) \begin{array}{l} \text{The annulus } 1 \leq r \leq 2 \\ \text{with } u(1, \theta) = 1 \text{ and } u(2, \theta) = 0. \end{array}$$