MATH 3363 FINAL EXAM. Sanders Fall 2006

This exam has 10 problems and all 10 problems will be graded. You have the full three hours to complete it. Use my supplied paper only and return your solution sheets with the problems in order. Put your name, **last name first please**, and **student id number** on each solution sheet you turn in. Good luck.

1. For the differential operator $\mathcal{L}(u) = \dfrac{d^2u}{dx^2}$, list only all eigenvalues and eigenfunctions for $\mathcal{L}$ subject to the following boundary conditions.

(a) $u(0) = 0 \;\; u(1) = 0$ (c) $u(0) = 0 \;\; u_x(1) = 0$

(b) $u_x(0) = 0 \;\; u_x(1) = 0$ (d) $u(-\pi) = u(\pi) \;\; u_x(-\pi) = u_x(\pi)$

2. The 2-π periodic Fourier series has the form $a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$. Expand f into such a series when:

(a) $f(x) = \cos^2(x) - \sin^2(x)$ (b) $f(x) = x$

(c) Use Parseval together with part (b) to show $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$. (10 points)

3. Solve the one dimensional heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

with periodic boundary conditions $\begin{array}{c} u(-\pi,t) = u(\pi,t) \\ u_x(-\pi,t) = u_x(\pi,t), \end{array}$

and initial condition $u(x,0) = x$.

Use your result in 2(b).

4. Solve the one dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

with boundary conditions $u(0,t) = 0, \;\; u_x(1,t) = 0,$

and initial conditions $\begin{array}{l} u(x,0) = \sin(3\pi x/2), \\ u_t(x,0) = \sin(\pi x/2). \end{array}$

5. Consider the inhomogeneous, one dimensional heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

with boundary conditions: $u(0,t) = 1, \;\; u(1,t) = 0,$
and initial condition: $u(x,0) = -x$.

(a) Determine the steady-state solution. (5 pts)

(b) Solve for $u(x,t)$. (15 pts)

6. Solve the following on the unit square, $0 < x < 1,\ 0 < y < 1$, subject to boundary conditions $u(x,0) = 0,\ u(x,1) = 0,\ u(0,y) = 0,\ u(1,y) = 0$.

(a) $\dfrac{\partial u}{\partial t} = \nabla^2 u$, with $u(x,y,0) = \sin(2\pi x)\sin(2\pi y)$.

(b) $\dfrac{\partial^2 u}{\partial t^2} = \nabla^2 u$, with $\begin{array}{l} u(x,y,0) = \sin(2\pi x)\sin(2\pi y) \\ u_t(x,y,0) = 0. \end{array}$

7. Solve Laplace's equation $\nabla^2 u = 0$ on the unit square $0 < x < 1,\ 0 < y < 1$ subject to the following boundary conditions.

(a) $\begin{array}{ll} u(x,0) = 0 & u(x,1) = 0 \\ u(0,y) = \sin(\pi y) & u(1,y) = 0 \end{array}$ (b) $\begin{array}{ll} u(x,0) = 0 & u(x,1) = \sin(\pi x) \\ u(0,y) = 0 & u(1,y) = 0 \end{array}$

8. Consider the one space dimensional wave equation on an infinite interval

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},\ t \geq 0,\ -\infty < x < \infty,$$

with initial conditions $u(x,0) = u_0(x)$, $u_t(x,0) = u_1(x)$.

Use the Fourier transform method to derive D'Alembert's solution

$$u(x,t) = \frac{1}{2}\left(u_0(x+t) + u_0(x-t) + \int_{x-t}^{x+t} u_1(y)\,dy\right).$$

9. Find the solution $u = u(r,\theta)$ of Laplace's equation $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0$ on the following domains satisfying the given boundary conditions.

(a) The unit disk $r < 1$ with $u(1,\theta) = 1 + \cos(2\theta)$.

(a) The annulus $1 < r < 2$ with $u(1,\theta) = 1$, and $u(2,\theta) = 0$.

10. Define the differential operator $\mathcal{L}(u) = \dfrac{d^2 u}{dr^2} + \dfrac{1}{r}\dfrac{du}{dr}$.

(a) Given that $u(1) = v(1) = 0$, prove that $\int_0^1 (u\mathcal{L}(v) - v\mathcal{L}(u))\,r\,dr = 0$. (Use integration by parts.)

(b) If $\mathcal{L}(u) = \lambda_1 u$ and $\mathcal{L}(v) = \lambda_2 v$ where $\lambda_1 \neq \lambda_2$, prove that $\int_0^1 u(r)v(r)\,r\,dr = 0$. (You may assume the result of the previous part.)