(1)

Sanders
Exam I

1. Determine all evals/fncts to

$$\frac{d^2e}{dx^2} = \lambda e \qquad e_x(0) = 0 \quad e(1) = 0.$$

You may assume $\lambda \in \mathbb{R}$.

Try e^{rx} to get char poly $r^2 - \lambda = 0$

Case 1 $\lambda = \sigma^2 > 0$

Here $e(x)$ has the form

$$e(x) = c_1 \cosh(\sigma x) + c_2 \sinh(\sigma x)$$

B.C.s imply

$$0 = e_x(0) = \sigma(c_1 \sinh(0) + c_2 \cosh(0)) \Rightarrow c_2 = 0$$

$$0 = e(1) = (c_1 \cosh(\sigma) + c_2 \sinh(\sigma))$$

$$\Rightarrow c_1 \cosh(\sigma) = 0 \Rightarrow c_1 = 0$$

So, no eigenfunctions here.

Case 2 $\lambda = 0$

$$e(x) = c_1 + c_2 x$$

B.C.s imply

$$0 = e_x(0) = c_2 \quad \Rightarrow c_2 = 0$$

$$0 = e(1) = c_1 + c_2 \quad \Rightarrow c_1 = 0$$

So, no eigenfunctions here either.

Case 3 $\lambda = -\sigma^2 < 0$ (over)

② Here, $e(x)$ has the form

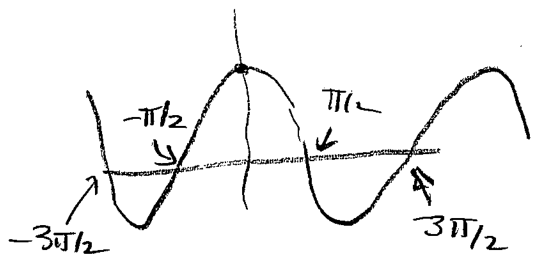
$$e(x) = c_1 \cos(\sigma x) + c_2 \sin(\sigma x)$$

B.C.'s imply

$$0 = e_x(0) = \sigma(-c_1 \sin 0 + c_2 \cos 0) \Rightarrow c_2 = 0$$
$$0 = e(1) = c_1 \cos\sigma + c_2 \sin\sigma \Rightarrow c_1 \cos\sigma = 0$$

But $\cos(\sigma) = 0$ when $\sigma = n\pi - \pi/2$ for any $n \in \mathbb{Z}$.

Let's call $e_n(x) = \cos((n-1/2)\pi x)$ and observe that



$$\begin{aligned}
e_{-n+1}(x) &= \cos((-n+1/2)\pi x) \\
&= \cos(-n\pi x)\cos(\pi/2\, x) - \sin(-n\pi x)\sin(\pi/2\, x) \\
&= \cos(n\pi x)\cos(\pi/2\, x) + \sin(n\pi x)\sin(\pi/2\, x) \\
&= \cos(n\pi x - \pi/2\, x) = \cos((n-1/2)\pi x) = e_n(x)
\end{aligned}$$

This tell us

$$e_0(x) = e_1(x)$$
$$e_{-1}(x) = e_2(x)$$
$$e_{-2}(x) = e_3(x)$$
$$\vdots$$

So we should enumerate $n = 1, 2, 3, \ldots$

$$\boxed{\begin{aligned} e_n(x) &= \cos((n-1/2)\pi x) \\ \lambda_n &= -((n-1/2)\pi)^2 \end{aligned}}$$

③

2a $\dfrac{d^2e}{dx^2} = \lambda e$

$e(1) = 0 \quad e(5) = 0$ $\quad 1 \quad x \quad 5 \qquad 0 \quad y \quad 1$

$$y(x) = \frac{(x-1)}{5-1} = \frac{1}{4}x - \frac{1}{4}$$

So, Let $y = \frac{1}{4}x - \frac{1}{4}$

$\Rightarrow \dfrac{d}{dx} = \dfrac{dy}{dx}\dfrac{d}{dy} = \dfrac{1}{4}\dfrac{d}{dy}$

$\Rightarrow \dfrac{d^2}{dx^2} = \dfrac{1}{4^2}\dfrac{d^2}{dy^2}$

Also let $\tilde{e}(y) = e(x)$

$\dfrac{d^2\tilde{e}}{dy^2} = \tilde{\lambda}\tilde{e} \quad (\tilde{\lambda} = 4^2\lambda)$

$\tilde{e}(0) = 0 \quad \tilde{e}(1) = 0$

So from given

$\tilde{e}_n(y) = \sin(n\pi y)$

$\tilde{\lambda}_n = -(n\pi)^2 \quad n = 1, 2 \ldots$

$$\boxed{\begin{aligned} e_n(x) &= \tilde{e}_n(y(x)) \\ &= \sin\left(n\pi\left(\tfrac{x-1}{4}\right)\right) \\ \lambda_n &= \tfrac{1}{4^2}\tilde{\lambda}_n = -\left(\tfrac{n\pi}{4}\right)^2 \end{aligned}}$$

for $n = 1, 2, \ldots$

b. $\dfrac{d^2e}{dx^2} = \lambda e \qquad e_x(-1) = 0$

$e_x(1) = 0$ $\quad -1 \quad x \quad 1 \qquad 0 \quad y \quad 1$

$$y(x) = \frac{x+1}{1-(-1)} = \frac{x+1}{2} = \frac{1}{2}x + \frac{1}{2}$$

Let $y = \frac{1}{2}x + \frac{1}{2} \Rightarrow \dfrac{d^2}{dx^2} = \dfrac{1}{2^2}\dfrac{d^2}{dy^2}$

Also let $\tilde{e}(y) = e(x)$

$\tilde{e}_x(-1) = \frac{1}{2}\tilde{e}_y(0) = 0$

$e_x(1) = \frac{1}{2}\tilde{e}_y(1) = 0$

$\dfrac{d^2\tilde{e}}{dy^2} = \tilde{\lambda}\tilde{e} \quad (\tilde{\lambda} = 2^2\lambda)$

$\tilde{e}_y(0) = 0 \quad \tilde{e}_y(1) = 0$

— So over —

(4)

$$\tilde{e}_n(y) = \cos(n\pi y)$$
$$\tilde{\lambda}_n = -(n\pi)^2 \quad n = 0, 1, \ldots$$

$$\boxed{\begin{aligned} e_n(x) &= \tilde{e}_n(y(x)) = \cos\left(n\pi\left(\tfrac{x+1}{2}\right)\right) \qquad n = 0, 1, 2 \ldots \\ \lambda_n &= \frac{1}{2^2}\tilde{\lambda}_n = -\left(\frac{n\pi}{2}\right)^2 \end{aligned}}$$

3 $\quad f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$ $\quad$ (FC) $\quad a_0 = \int_0^1 f(y)\,dy$

$$a_n = 2\int_0^1 f(y)\cos(n\pi y)\,dy$$

a. Derive the Fourier Coff formulae,

(*) $\displaystyle \int_0^1 f(x)\cos(m\pi x) = a_0\int_0^1 \cos(m\pi x) + \sum_{n=1}^{\infty} a_n \int_0^1 \cos(n\pi x)\cos(m\pi x)$

$$\cos(a)\cos(b) = \frac{1}{2}(\cos(a-b) + \cos(a+b))$$

So

$$\int_0^1 \cos(n\pi x)\cos(m\pi x)\,dx = \begin{cases} 1 & \text{if } m = n = 0 \\ \int_0^1 \left(1 + \frac{\cos(2n\pi x)}{2}\right) = \frac{1}{2} & \text{if } m = n > 0 \\ \frac{1}{2}\int_0^1 \cos((n-m)\pi x) + \cos((n+m)\pi x) & \\ = \frac{1}{2}\left(\frac{\sin((n-m)\pi x)}{(n-m)\pi} + \frac{\sin((n+m)\pi x)}{(n+m)\pi}\right)\Big|_0^1 & \\ = 0 & \text{if } m \neq n. \end{cases}$$

Use (*)

$m = 0$ $\quad \int_0^1 f(x) = a_0 + 0$

$m > 0$ $\quad \int_0^1 f(x)\cos(m\pi x)$

$$= 0 + \frac{1}{2}a_m$$

This gives (FC) above.

(5)

3b. $f(x) = \cos^2(\pi x) = \frac{1}{2}(1+\cos(2\pi x))$

$$= \boxed{\frac{1}{2} + \frac{1}{2}\cos(2\pi x)}$$

$$\begin{cases} a_0 = \frac{1}{2} \\ a_2 = 1/2 \\ \text{all other } a_n\text{'s} = 0 \end{cases}$$

3c $f(x) = x$ $\quad a_0 = \int_0^1 x\,dx = 1/2$

$$a_n = 2\int_0^1 \overset{u}{x} \overset{dv}{\cos(n\pi x)\,dx}$$

$$= 2\left[x\frac{\sin(n\pi x)}{n\pi}\Big|_0^1 - \int_0^1 \frac{\sin(n\pi x)}{n\pi}dx \right]$$

$$= 2\left[\frac{\cos(n\pi x)}{(n\pi)^2} \right|_0^1 = \frac{2(\cos(n\pi)-1)}{(n\pi)^2}$$

$$\boxed{x \sim \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2(\cos(n\pi)-1)}{(n\pi)^2}\cos(n\pi x)}$$

3c $f(x) = x + 1 \sim \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2(\cos(n\pi)-1)}{(n\pi)^2}\cos(n\pi x) + 1$

$$\boxed{x+1 \sim \frac{3}{2} + \sum_{n=1}^{\infty} \frac{2(\cos(n\pi)-1)}{(n\pi)^2}\cos(n\pi x)}$$

⑥

4a $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ $\Rightarrow u(x,t) = \alpha_0(t) + \sum_{n=1}^{\infty} \alpha_n(t)\cos(n\pi x)$

$u_x(0,t) = 0 \quad u_x(1,t) = 0$

$u(x,0) = 2 + 3\cos(4\pi x)$

Plug into PDE to get $\frac{\partial \alpha_0}{\partial t} + \sum_{n=1}^{\infty} \frac{\partial \alpha_n}{\partial t}\cos(n\pi x)$

$= \alpha_0 \frac{\partial^2 1}{\partial x^2} + \sum_{n=1}^{\infty} \alpha_n \frac{\partial^2}{\partial x^2}\cos(n\pi x)$

$= \alpha_0 \cdot 0 + \sum_1 -(n\pi)^2 \alpha_n \cos(n\pi x)$

$\frac{d\alpha_0}{dt} = 0$

$\frac{d\alpha_n}{dt} = -(n\pi)^2 \alpha_n$ for $n = 1, 2 \ldots$ $\Leftarrow$

$\rightarrow \alpha_0(t) = a_0$

$\alpha_n(t) = a_n e^{-(n\pi)^2 t}$ for $n = 1, 2 \ldots$

So $u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-(n\pi)^2 t}\cos(n\pi x)$

$2 + 3\cos(4\pi x) = u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$

$\Rightarrow a_0 = 2 \quad a_4 = 3$ all other a_n's $= 0$.

$$\boxed{u(x,t) = 2 + 3e^{-(4\pi)^2 t}\cos(4\pi x)}$$

(7)

4b $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

$u_x(0,t) = 0 \quad u_x(1,t) = 0$

$u(x,0) = x$

From part a $\quad u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-(n\pi)^2 t} \cos(n\pi x)$

From (3b)

$$x = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2(\cos(n\pi) - 1)}{(n\pi)^2} \cos(n\pi x)$$

$$= u(x,0) = a_0 + \sum_{n} a_n \cos(n\pi x)$$

$$\Rightarrow a_0 = 1/2 \qquad a_n = \frac{2(\cos(n\pi) - 1)}{(n\pi)^2} \quad n = 1, 2 \ldots$$

$$\boxed{u(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2(\cos(n\pi) - 1)}{(n\pi)^2} e^{-(n\pi)^2 t} \cos(n\pi x)}$$

(8)

5a $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ $\Rightarrow$ $u(x,t) = \alpha_0(t) + \sum_{n=1}^{\infty} \alpha_n(t)\cos(n\pi x)$

$u_x(0,t)=0 \quad u_x(1,t)=0$

(I.C) $\begin{cases} u(x,0)=0 \\ u_t(x,0) = 2+3\cos(4\pi x) \end{cases}$

plug into PDE to get $\frac{\partial^2 \alpha_0}{\partial t^2} + \sum_{n=1}^{\infty} \frac{\partial^2 \alpha_n}{\partial t^2}\cos(n\pi x)$

$= \alpha_0 \frac{\partial^2 1}{\partial x^2} + \sum_{n=1}^{\infty} \alpha_n \frac{\partial^2}{\partial x^2}\cos(n\pi x)$

$= \alpha_0 \cdot 0 + \sum_{n=1}^{\infty} -(n\pi)^2 \alpha_n \cos(n\pi x)$

$\Leftarrow$

$\frac{d^2\alpha_0}{dt^2} = 0$

$\frac{d^2\alpha_n}{dt^2} + (n\pi)^2\alpha_n = 0 \quad n=1,2,\ldots$

$\Rightarrow \alpha_0(t) = a_0 + b_0 t$

$\alpha_n(t) = a_n\cos(n\pi t) + b_n \sin(n\pi t)$

$u(x,t) = (a_0 + b_0 t) + \sum_{n=1}^{\infty}\left(a_n\cos(n\pi t) + b_n \sin(n\pi t)\right)\cos(n\pi x)$

use (I.C.)

$0 = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$

$2+3\cos(4\pi x) = u_t(x,0) = b_0 + \sum_{n=1}^{\infty}(n\pi)\, b_n \cos(n\pi x)$

$b_0 = 2$ $\quad b_4 = \frac{3}{4\pi}$ $\quad$ all other $a_n=0$, $b_n=0$ $\Rightarrow$ $\boxed{u(x,t) = 2t + \frac{3}{4\pi}\sin(4\pi t)\cos(4\pi x)}$

⑨

5b. $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$

$u_x(0,t) = 0 \quad u_x(1,t) = 0$

(I.C.) $\left| \begin{array}{l} u(x,0) = x \\ u_t(x,0) = 0 \end{array} \right.$

As done in part (a)

$$u(x,t) = (a_0 + b_0 t) + \sum_{n=1}^{\infty} \left(a_n \cos(n\pi t) + b_n \sin(n\pi t) \right) \cos(n\pi x)$$

use (I, C.)

$$x \quad (3c) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2(\cos(n\pi) - 1)}{(n\pi)^2} \cos(n\pi x)$$

$$= u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

$$0 = u_t(x,0) = b_0 + \sum_{n=1}^{\infty} n\pi b_n \cos(n\pi x)$$

$$\Rightarrow a_0 = 1/2, \quad a_n = \frac{2(\cos(n\pi) - 1)}{(n\pi)^2} \quad n = 1, 2 \cdots$$

and all b_n's $= 0$.

So

$$\boxed{u(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2(\cos(n\pi) - 1)}{(n\pi)^2} \cos(n\pi t) \cos(n\pi x)}$$