$$\frac{d}{dx} = \frac{dy}{dx}\frac{d}{dy}$$

$y = y(x)$

Q1

$$\frac{d}{dx}\left(\frac{d}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dx}\frac{d}{dy}\right)$$

$$\frac{d^2}{dx^2} = \frac{dy}{dx}\frac{d}{dx}\left(\frac{d}{dy}\right) + \frac{d^2y}{dx^2}\frac{d}{dy}$$

$$= \frac{dy}{dx}\frac{dy}{dx}\frac{d}{dy}\left(\frac{d}{dy}\right) + \frac{d^2y}{dx^2}\frac{d}{dy}$$

$$= \left(\frac{dy}{dx}\right)^2\frac{d^2}{dy^2} + \frac{d^2y}{dx^2}\frac{d}{dy}$$

In most examples we took

$y = \alpha x + \beta$ α, β constants.

$$\frac{dy}{dx} = \alpha, \quad \frac{d^2y}{dx^2} = 0$$

$$\frac{d^2}{dx^2} = \alpha^2\frac{d^2}{dy^2} + 0$$

Q2.1

Green's Ident,

$$\int_a^b \left(u \frac{d^2 v}{dx^2} - v \frac{d^2 u}{dx^2} \right) dx$$

I by Parts

$$= \left(u \frac{dv}{dx} - v \frac{du}{dx} \right) \Big|_a^b$$

$$\boxed{\int_a^b f \frac{dg}{dx} = fg \Big|_a^b - \int_a^b \frac{df}{dx} g}$$

$$- \int_a^b \left(\frac{du}{dx}\frac{dv}{dx} - \frac{dv}{dx}\frac{du}{dx} \right) dx$$

$$= \left(u \frac{dv}{dx} - v \frac{du}{dx} \right) \Big|_a^b - 0 .$$

Use this to show evals for

$$\begin{cases} \dfrac{d^2 e}{dx^2} = \lambda e \\ e'(0) = 0, \; e'(1) = 0 \end{cases}$$

must be real.

Q2.2

Assume λ complex e complex.

$\overline{\frac{d}{dx} e} = \frac{d}{dx}\bar{e}$, $\quad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

$\bar{z} z = |z|^2 \qquad z = x + iy$

$\quad \| \qquad\qquad \bar{z} = x - iy$

$z\bar{z}$

$\bar{z} z = x^2 + y^2$

$\bar{z} z = |z|^2 \geqslant 0 \quad |z|^2 = 0 \iff z = 0.$

$$\boxed{\bar{e} \frac{d^2 e}{dx^2} = \bar{e} \lambda e} = \lambda |e|^2$$

$$\overline{\frac{d^2 e}{dx^2}} = \overline{\lambda e}$$

$$\boxed{e \frac{d^2 \bar{e}}{dx^2} = e \bar{\lambda} \bar{e}} = \bar{\lambda} |e|^2$$

Q2, 3

$$\int_0^1 \left(\bar{e}\frac{d^2 e}{dx^2} - e\frac{d^2\bar{e}}{dx^2}\right) = \int_0^1 (\lambda-\bar{\lambda})|e|^2$$

$$= (\lambda-\bar{\lambda})\int_0^1 |e|^2$$

Green's

LHS

$$= \left(\bar{e}\frac{de}{dx} - e\frac{d\bar{e}}{dx}\right)\Bigg|_0^1$$

$$\boxed{\begin{array}{l} \text{B. C.} \\ e'(0)=0 \\ \bar{e}'(0)=0 \\ e'(1)=0 \\ \bar{e}'(1)=0 \end{array}}$$

$$= \left(\bar{e}(1)\cancel{\frac{de}{dx}(1)}^{0} - e(1)\cancel{\frac{d\bar{e}}{dx}(1)}^{0}\right)$$

$$- \left(\bar{e}(0)\cancel{\frac{d}{dx}e(0)}^{0} - e(0)\cancel{\frac{d}{dx}\bar{e}(0)}^{0}\right)$$

$$= 0$$

not zero since e is a nontrivial function

$$\therefore (\lambda-\bar{\lambda})\int_0^1 |e|^2 = 0$$

$$\Rightarrow \lambda-\bar{\lambda}=0$$

Can you use the evals and efncts to (Q3.1)

$$\frac{d^2e}{dx^2} = \lambda e \qquad e(-\pi) = e(\pi), \quad e'(-\pi) = e'(\pi)$$

to find the evals and efncts to

$$\frac{d^2e}{dx^2} = \lambda e \qquad e(0) = e(1), \quad e'(0) = e'(1) \quad ?$$

Should have memorized that

$$\frac{d^2e}{dx^2} = \lambda e$$

$$e(-\pi) = e(\pi)$$

$$e'(-\pi) = e'(\pi)$$

$$\begin{cases} \lambda_0 = 0, \quad e_0(x) = 1 \\ n = 1, 2, \ldots \\ \lambda_n = -n^2 \\ u_n(x) = \cos(nx) \\ u_n(x) = \sin(nx) \end{cases}$$

(Q3.2

Transform the interval

$(0, 1) \xrightarrow{y(x)} (-\pi, \pi)$

$y(x) = (2\pi)(x-0) - \pi$

$\frac{d^2}{dx^2} e(x) = \lambda e(x)$ $f(y) = e(x)$

$(2\pi)^2 \frac{d^2}{dy^2} f(y) = \lambda f(y)$ $f(-\pi) = e(0) = e(1) = f(\pi)$ etc.

$f(-\pi) = f(\pi)$

$f'(-\pi) = f'(\pi)$

$\frac{d^2 f}{dy^2} = \frac{1}{(2\pi)^2} \lambda f \equiv \tilde{\lambda} f$

Use this and what we know already

$\tilde{\lambda}_0 = 0$ $f_0(y) = 1$

$n = 1, 2, \ldots$ $\tilde{\lambda}_n = -n^2$ $f_n(y) = \cos(ny)$, $f_n(y) = \sin(ny)$

But I want λ (not $\tilde{\lambda}$) and $e(x)$ (not $f(y)$)

Q3,3

$$\begin{cases} \lambda_0 = (2\pi)^2 \tilde{\lambda}_0 = 0 \\ e_0(x) = f_0(y) = 1 \end{cases}$$

$$\lambda_n = (2\pi)^2 \tilde{\lambda}_n = -(2\pi)^2 n^2$$

$$e_n(x) = f_n(y) = \cos(ny)$$
$$= \cos(n(2\pi x - \pi))$$
$$= (-1)^n \cos(2n\pi x)$$

$$\left(\cos(2n\pi x - n\pi) = \cos(2n\pi x)\cos(-n\pi) - \sin(2n\pi x)\underbrace{\sin(-n\pi)}_{0} \right.$$

$\cos(n\pi) = \pm 1$

$$= (-1)^n \cos(2n\pi x)$$

Similarly

$$e_n(x) = \sin(ny) = \sin(n(2\pi x - \pi))$$
$$= (-1)^n \sin(2n\pi x)$$

$$\begin{cases} n = 1, 2, 3, \cdots \\ \lambda_n = -(2n\pi)^2 \\ e_n(x) = \cos(2n\pi x),\ e_n(x) = \sin(2n\pi x) \end{cases}$$

(Q3.4

From Scratch (Can also do this from scratch)

$$\frac{d^2 e}{dx^2} = \lambda e \qquad \begin{cases} e(0) = e(1) \\ e'(0) = e'(1) \end{cases}$$

$r^2 - \lambda = 0$

$\lambda = \sigma^2 > 0$ — ✓ do this —

$\lambda = 0 \qquad e(x) = 1$

$\lambda = -\sigma^2 < 0$

$e(x) = c_1 \cos(\sigma x) + c_2 \sin(\sigma x)$

$c_1 = c_1 \cos(\sigma) + c_2 \sin(\sigma)$

$c_2 = -c_1 \sin(\sigma) + c_2 \cos(\sigma)$

$(\cos(\sigma) - 1) c_1 + \sin(\sigma) c_2 = 0$

$-\sin(\sigma) c_1 + (\cos(\sigma) - 1) c_2 = 0$

$$\begin{pmatrix} \cos(\sigma)-1 & \sin\sigma \\ -\sin\sigma & \cos\sigma-1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (43\text{c})$$

has non trivial $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \Leftrightarrow \det(M)=0$

$$\det\begin{pmatrix} \cos\sigma-1 & \sin\sigma \\ -\sin\sigma & \cos\sigma-1 \end{pmatrix}$$

$$= (\cos\sigma-1)^2 + \sin^2\sigma = 0$$

$$\cos^2\sigma - 2\cos\sigma + 1 + \sin^2\sigma$$

$$= 2-2\cos\sigma = 2(1-\cos\sigma)=0$$

$$\Rightarrow \cos\sigma = 1$$

$$\sigma = 2n\pi \qquad n=1,2,---$$

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ c_1 & c_2 can be anything!

(Q3.6

get two eigen functions
indep

$\lambda_n = -\sigma^2 = -(2n\pi)^2$

$e_n(x) = \cos(\sigma x) = \cos(2n\pi x)$

$e_n(x) = \sin(\sigma x) = \sin(2n\pi x)$

$n = 1, 2, \ldots$

See — same as before.