Q1.1

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$u(0,t)=0 \quad u_x(1,t)=0$

$u(x,0) = 9 + 43\cos(12\pi x)$

$$\frac{d^2 e}{dx^2} = \lambda e$$

$e'(0)=0 \quad e'(1)=0$

$\lambda_0 = 0 \quad e_0(x) = 1$ all efncts s'vcls

$\lambda_n = -(n\pi)^2 \quad e_n(x) = \cos(n\pi x) \quad n = 1, 2 \ldots$

$$u(x,t) = \sum_{n=0}^{\infty} \alpha_n(t)\cos(n\pi x)$$

$\alpha_n'(t)$

$$\sum_{n=0}^{\infty}\left(\frac{\partial \alpha_n}{\partial t}\right)\cos(n\pi x) = \sum_{n=0}^{\infty}\left(-(n\pi)^2\alpha_n\right)\cos(n\pi x)$$

LHS $\quad \frac{\partial \alpha_n}{\partial t} = -(n\pi)^2\alpha_n \qquad \alpha_n(t) = c_n e^{-(n\pi)^2 t}$

ODE

Suppose I know the soln to Q2.1

$$(*)\begin{cases}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 & -\infty < x < \infty \\ u(x,0) = f(x) & u(x,t) = \frac{1}{2}(f(x+t) + f(x-t)) \\ u_t(x,0) = 0 & \end{cases}$$

Can I use above knowledge to solve

$$(**)\begin{cases}\frac{\partial^2 u}{\partial t^2} - 12\frac{\partial^2 u}{\partial x^2} = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = 0\end{cases}$$

?

Yes by changing variables.

$$x = \sqrt{12}\, y \qquad \boxed{\frac{\partial}{\partial x} = \frac{1}{\sqrt{12}}\frac{\partial}{\partial y}}$$

$$\frac{\partial}{\partial x} = \frac{\partial y}{\partial x}\frac{\partial}{\partial y} = \frac{1}{\sqrt{12}}\frac{\partial}{\partial y}$$

$$\frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{12}}\frac{\partial}{\partial y}\right) = \frac{1}{\sqrt{12}}\frac{1}{\sqrt{12}}\frac{\partial^2 u}{\partial y^2}$$

$$\boxed{\frac{\partial^2}{\partial x^2} = \frac{1}{12}\frac{\partial^2}{\partial y^2}}$$

Q2.2

$u(x,t) = v(y(x),t) = v(y,t)$

$$\begin{cases} \dfrac{\partial^2 v}{\partial t^2} - \dfrac{\cancel{12}}{\cancel{12}} \dfrac{\partial^2 v}{\partial y^2} = 0 \\ v(y,0) = f(\sqrt{12}\, y) \\ v_t(y,0) = 0 \end{cases}$$

looks like $(*)$

$$v(y,t) = \frac{1}{2}\Big(f(\sqrt{12}(y+t)) + f(\sqrt{12}(y-t)) \Big)$$

$\|$

$$u(x,t) = v(y(x),t)$$

$$= \frac{1}{2}\Big(f\big(\sqrt{12}\big(\tfrac{1}{\sqrt{12}}x+t\big)\big) + f\big(\sqrt{12}\big(\tfrac{1}{\sqrt{12}}x-t\big)\big) \Big)$$

$$= \frac{1}{2}\Big(f(x+\sqrt{12}\,t) + f(x-\sqrt{12}\,t) \Big)$$

This solves $(**)$

know on
$[-\pi, \pi]$

Solve on Q3.1
$[0, 1]$

y

$[-\pi, \pi] \xleftarrow{y(x)} [0, 1]$

$y(0) = -\pi$
$y(1) = \pi$

$$\boxed{y(x) = 2\pi(x-0) - \pi}$$

$$\frac{d^2 y}{dx^2} = \lambda u \qquad x \in [0, 1]$$

$u(x) = v(y(x))$

chain rule

$$\frac{d}{dx} = \frac{dy}{dx}\frac{d}{dy} = 2\pi \frac{d}{dy}$$

$$\frac{d^2}{dx^2} = (2\pi)^2 \frac{d^2}{dy^2} \qquad y \in [-\pi, \pi]$$

$v(y) = u(x)$

$$(2\pi)^2 \frac{d^2 v}{dy^2} = \lambda v \qquad \frac{d^2 v}{dy^2} = \tilde{\lambda} v \qquad \tilde{\lambda} = \frac{\lambda}{(2\pi)^2}$$

I can solve.