Q1.1

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad u(x,t) = \sum_{n=1}^{\infty} \alpha_n(t) \sin(n\pi x)$$

$u(0,t) = 0, \; u(1,t) = 0$ B.C. → e funct prob #1

For u to satisfy PDE

$n = 1, 2, \ldots \quad \frac{\partial \alpha_n}{\partial t} = -(n\pi)^2 \alpha_n$ constants.

$$\Rightarrow \alpha_n(t) = e^{-(n\pi)^2 t} a_n$$

i.e. Soln has the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-(n\pi)^2 t} \sin(n\pi x)$$

I.C. $\quad u(x,0) = f(x)$ Gotta determine the constants a_1 a_2 --- to sat $u(x,0) = f(x)$

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$$

$$\boxed{a_n = 2\int_0^1 f(y) \sin(n\pi y)\, dy}$$

Q1.2

$$\frac{\partial}{\partial t}\sum_{n=1}^{\infty}\alpha_n(t)\sin(n\pi x) = \frac{\partial^2}{\partial x^2}\sum_{n=1}^{\infty}\alpha_n(t)\sin(n\pi x)$$

$$\sum \frac{\partial \alpha_n(t)}{\partial t}\sin(n\pi x) = \sum_{n=1}^{\infty}\alpha_n(t)\left(-(n\pi)^2\right)\sin(n\pi x)$$

for all x, t

$$\frac{\partial \alpha_n}{\partial t} = -(n\pi)^2\alpha_n$$

for each $n = 1, 2, \ldots$

These are const coeff linear 1st order ODEs

$$\alpha_n(t) = e^{-(n\pi)^2 t} a_n$$

< 0

$$u(x,t) = \sum_{n=1}^{\infty}\left(2\int_0^1 f(y)\sin(n\pi y)\,dy\right)\underline{e^{-(n\pi)^2 t}}\sin(n\pi x)$$

tends to zero for each $n = 1$.

$\to 0$ as $t \to \infty$.

91.3

In class I did

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

to me to

$$u_x(0,t)=0 \quad u_x(1,t)=0$$

$$u(x,t) = \sum_{n=0}^{\infty} \alpha_n(t)\cos(n\pi x)$$

$$u(x,0)=f(x)$$

ef prob #II

like before etc. - -

$$f(x) = \left(\int_0^1 f(y)\,dy\right)\cdot 1 + \sum_{n=1}^{\infty}\left(2\int_0^1 f(y)\cos(n\pi y)\,dy\right)\cos(n\pi x)$$

$$\int_0^1 \cos(m\pi x)\,f(x) = \sum_{n=0}^{\infty} a_n \int_0^1 \cos(n\pi x)\cos(m\pi x) = a_m\int_0^1 \cos^2(m\pi x)\,dx$$

$$= a_m \begin{cases} 1 & m=0 \\ 1/2 & m>0 \end{cases}$$

$$\int_0^1 \cos(n\pi x)\cos(m\pi x)\,dx = \begin{cases} 0 & n \neq m \\ 1/2 & n=m>0 \\ 1 & n=m=0 \end{cases}$$

(2) $\int_0^1 f(x)\cos(m\pi x)\,dx \overset{m>0}{=} a_m$

(1) $\int_0^1 f(x)\cos(m\pi x)\,dx \quad m=0$

(Q1.4

$$u(x,t) = \int_0^1 f(y)\,dy \cdot 1 + \sum_{n=1}^{\infty} \left(2\int_0^1 f(y)\cos n\pi y\,dy\right) e^{-(n\pi)^2 t} \cos(n\pi x)$$

$\alpha_n(t) = a_n e^{-(n\pi)^2 t}$

as $t \to \infty$ note that each term in here tends to zero

$$u(x,t) \to \int_0^1 f(y)\,dy$$

Q2

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$$

$$a_n = 2\int_0^1 f(y)\sin(n\pi y)\,dy$$

$$f(x) = \sum_{n=0}^{\infty} a_n \cos(n\pi x)$$

$$a_n = \begin{cases} \int_0^1 f(y)\,dy & n=0 \\ 2\int_0^1 f(y)\cos(n\pi y) & n>0 \end{cases}$$

III, IV you do These.

V $$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

$$a_0 = \frac{1}{2\pi}\int_0^1 f(y)\,dy \quad a_n = \frac{1}{\pi}\int_1^1 f(y)\cos(ny)\,dy$$

$$b_n = \frac{1}{\pi}\int_1^1 f(y)\sin(ny)\,dy$$

Q3.1

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$\left.\begin{aligned} u(-\pi, t) &= u(\pi, t) \\ u_x(-\pi, t) &= u_x(\pi, t) \end{aligned}\right\} B.C$$

$$u(x, 0) = |x|$$

tells us to look for a sol'n

$$u(x,t) = \alpha_0(t) + \sum_{n=1}^{\infty} \alpha_n(t)\cos(nx) + \beta_n(t)\sin(nx)$$

plug in to PDE

$$\frac{d\alpha_0}{dt} = 0$$

$$\left.\begin{aligned} \frac{d\alpha_n}{dt} &= -n^2 \alpha_n \\ \frac{d\beta_n}{dt} &= -n^2 \beta_n \end{aligned}\right\} n = 1, 2, \ldots$$

Q3.2

$$\frac{\partial}{\partial t}\left(\alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos(nx) + \beta_n \sin nx\right)$$

$$= \frac{\partial^2}{\partial x^2}\left(\alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos nx + \beta_n \sin(nx)\right)$$

$$LHS = \frac{\partial \alpha_0}{\partial t} + \sum_{n=1}^{\infty}\left(\frac{\partial \alpha_n}{\partial t}\cos(nx) + \frac{\partial \beta_n}{\partial t}\sin(nx)\right)$$

$$\updownarrow$$

$$RHS = 0 + \sum_{n=1}^{\infty} -n^2 \alpha_n \cos(nx) - n^2 \beta_n \sin(nx)$$

$$\frac{d\alpha_0}{dt} = 0 \qquad \alpha_0 = a_0$$

$$\frac{d\alpha_n}{dt} = -n^2 \alpha_n \qquad \alpha_n(t) = a_n e^{-n^2 t}$$

$$\frac{d\beta_n}{dt} = -n^2 \beta_n \qquad \beta_n = b_n e^{-n^2 t}$$

(Q3.3

$u(x,t)$

$$= a_0 + \sum_{n=1}^{\infty} e^{-n^2 t}\left(a_n \cos(nx) + b_n \sin(nx)\right)$$

$u(x,0) = |x|$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin(nx)$$

From earlier exercise

$a_0 = \pi/2 \quad a_n = \frac{2(\cos(n\pi)-1)}{n^2\pi} \quad n=1,2\ldots$

$b_n = 0 \quad \forall n.$

Q4,1

$$\boxed{\frac{d^2e}{dx^2} - 2\frac{de}{dx} = \lambda e \qquad u(0)=0 \qquad u(1)=0}$$

$e_n(x) = e^x \sin(n\pi x) \quad n=1,2,3\ldots$

$\lambda_n - (n\pi)^2$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x} \qquad \left(\frac{\partial^2}{\partial x^2} - 2\frac{\partial}{\partial x}\right)u$$

$u(0,t)=0 \qquad u(1,t)=0$

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n(t)\, e_n(x) \qquad \left(\frac{\partial^2}{\partial x^2} - 2\frac{\partial}{\partial x}\right)e_n = \lambda_n e_n$$

$$LHS = \sum_{n=1}^{\infty} \frac{\partial \alpha_n}{\partial t}\left(e^x \sin(n\pi x)\right)$$

$$RHS = \sum_{n=1}^{\infty} -(n\pi)^2 \alpha_n \left(e^x \sin(n\pi x)\right)$$

Just like before §4.2

$$\frac{\partial}{\partial t}\alpha_n = -(n\pi)^2\alpha_n$$

$$\cancel{u(x,t)}\quad \alpha_n(t) = a_n e^{-(n\pi)^2 t}$$

$$u(x,t) = \sum_{n=1}^{\infty} \underline{\underline{a_n}}\, e^{-(n\pi)^2 t} \underbrace{e^x \sin(n\pi x)}_{e_n(x)}$$

The I.C.'s will determine the a_n's.