Solve

Q1.1

$$\frac{\partial^2 u}{\partial t^2} - u = \frac{\partial^2 u}{\partial x^2}$$

$$\left.\begin{aligned} u(-\pi,t) &= u(\pi,t) \\ u_x(-\pi,t) &= u_x(\pi,t) \end{aligned}\right\} B.C. \qquad \left.\begin{aligned} u(x,0) &= |x| \\ u_t(x,0) &= 0 \end{aligned}\right\} I.C.$$

$$u(x,t) = \alpha_0(t) + \sum_{n=1}^{\infty}\left(\alpha_n(t)\cos(nx) + \beta_n(t)\sin(nx)\right)$$

$$\frac{d^2\alpha_0}{dt^2} - \alpha_0 = -0^2\alpha_0$$

$$\frac{d^2\alpha_n}{dt^2} - \alpha_n = -n^2\alpha_n \qquad \text{note when } n=1 \quad \frac{d^2\alpha_1}{dt^2} = 0$$

$$\frac{d^2\beta_n}{dt^2} - \beta_n = -n^2\beta_n \qquad \text{when } n=1 \quad \frac{d^2\beta_1}{dt^2} = 0$$

$$\Rightarrow \quad \alpha_0(t) = a_0\cosh t + \tilde{a}_0 \sinh t$$

$$\alpha_1(t) = a_1 + \tilde{a}_1 t$$

$$n \geq 2 \quad \alpha_n(t) = a_n\cos(\sqrt{n^2-1}\,t) + \tilde{a}_n\sin(\sqrt{n^2-1}\,t)$$

$$\beta_1(t) = b_1 + \tilde{b}_1 t$$

$$n \geq 2 \quad \beta_n(t) = b_n\cos(\sqrt{n^2-1}\,t) + \tilde{b}_n\sin(\sqrt{n^2-1}\,t)$$

§1.2

*

$$u(x,t) = (a_0 \cosh t + \tilde{a}_0 \sinh t) \cdot 1$$
$$+ (a_1 + \tilde{a}_1 t)\cos(1x) + (b_1 + \tilde{b}_1 t)\sin(1x)$$
$$+ \sum_{n=2}^{\infty} \Big[\big(a_n \cos(\sqrt{n^2-1}\,t) + \tilde{a}_n \sin(\sqrt{n^2-1}\,t)\big)\cos(nx)$$
$$+ \big(b_n \cos(\sqrt{n^2-1}\,t) + \tilde{b}_n \sin(\sqrt{n^2-1}\,t)\big)\sin(nx) \Big]$$

Now to eval all these constants.

$\cos(n\pi) = (-1)^n$

$$u(x,0) = |x| = \frac{\pi}{2} + \frac{-4}{\pi}\cos(1x) + \sum_{n=2}^{\infty} \frac{2((-1)^n - 1)}{n^2 \pi}\cos(nx)$$

$$u_t(x,0) = 0 = 0 + 0\cos(1x) + \sum_{n=2}^{\infty} 0 \cos(nx)$$

$$u(x,0) = a_0 + a_1 \cos x + b_1 \sin x + \sum_{n=2}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

$$u_t(x,0) = \tilde{a}_0 + \tilde{a}_1 \cos(x) + \tilde{b}_1 \sin(x) + \sum_{n=2}^{\infty} \sqrt{n^2-1}\,\tilde{a}_n \cos(nx) + \sqrt{n^2-1}\,\tilde{b}_n \sin(nx)$$

So Let's match up terms.

Q1.3

$a_0 = \pi/2$

$a_1 = -\frac{4}{\pi}$ $\qquad b_1 = 0$

$n \geq 2 \quad a_n = \frac{2((-1)^n - 1)}{n^2 \pi} \qquad b_n = 0 \quad n \geq 2$

$\tilde{a}_0 = 0$

$\tilde{a}_1 = 0 \qquad \tilde{b}_1 = 0$

$\forall n \geq 2 \quad \tilde{a}_n = 0 \qquad \tilde{b}_n = 0$

Plug this into (*)

$$u(x,t) = \frac{\pi}{2}\cosh t - \frac{4}{\pi}\cos x + \sum_{n=2}^{\infty} \frac{2((-1)^n-1)}{n^2\pi}\cos(\sqrt{n^2-1}\,t) \cdot \cos(nx)$$

Simplify somewhat

$$\boxed{u(x,t) = \frac{\pi}{2}\cosh t + \sum_{n=1}^{\infty} \frac{2((-1)^n-1)}{n^2\pi}\cos(\sqrt{n^2-1}\,t)\cos(nx)}$$

Heat Equation:

Q2.1

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$\left.\begin{array}{l} u(0,t)=0 \\ u(1,t)=0 \end{array}\right\} \text{B.C.} \qquad u(x,0)= \sin(3\pi x)+19\sin(4\pi x) \Big\} \text{I.C.}$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} \; ; \; u(0)=0 \quad u(1)=0}$$ tells us to use eigenfunction expansion

$$(*) \quad u(x,t)=\sum_{n=1}^{\infty} \alpha_n(t) \sin(n\pi x)$$

Plug in to PDE

$$\frac{\partial}{\partial t}\sum_{n=1}^{\infty} \alpha_n \sin(n\pi x) = \frac{\partial^2}{\partial x^2}\sum_{n=1} \alpha_n \sin(n\pi x)$$

$$\sum_{n=1}\left(\frac{\partial \alpha_n}{\partial t}\right)\sin(n\pi x) = \sum_{n=1}^{\infty} \alpha_n\left(-(n\pi)^2 \sin(n\pi x)\right)$$

for each $n=1,2,\dots$

$$\frac{d\alpha_n}{dt} = -(n\pi)^2 \alpha_n \Rightarrow \alpha_n(t) = a_n e^{-(n\pi)^2 t}$$

$$r+(n\pi)^2=0$$

Plug these back into $*$

over

Q2.2

(##)
$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-(n\pi)^2 t} \sin(n\pi x)$$

$$u(x,0) = \sin(3\pi x) + 19\sin(4\pi x)$$

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$$

I could write

$$= 0\cdot\sin(1\pi x) + 0\cdot\sin(2\pi x) + 1\sin(3\pi x)$$
$$+ 19\sin(4\pi x) + 0\sin(5x) + 0\sin(6\pi x) + \cdots$$

$a_1 = 0 \quad a_2 = 0$

$a_3 = 1, \quad a_4 = 19, \quad a_5 = 0, \ldots$

plug these into (**) to get

$$\boxed{u(x,t) = 1\cdot e^{-(3\pi)^2 t}\sin(3\pi x) + 19 e^{-(4\pi)^2 t}\sin(4\pi x)}$$

(Q3,1)

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} \\ u_x(0,t)=0 \\ u_x(1,t)=0 \end{cases} \quad \Big\} \text{B.C.}$$

$u(x,0) = 1 + 13\cos(5\pi x)$

Because of $\dfrac{\partial^2 u}{\partial x^2}$ & $u_x(0)=0$, $u_x(1)=0$ $\Rightarrow$ (*) $u(x,t) = \alpha_0(t) + \sum_{n=1}^{\infty} \alpha_n(t)\cos(n\pi x)$

plug back into PDE

$$\begin{cases} \dfrac{d\alpha_0}{dt} = 0 \\ \dfrac{d\alpha_n}{dt} = -(n\pi)^2 \alpha_n \qquad n=1,2,\ldots \end{cases}$$

$\alpha_0(t) = a_0 \qquad \alpha_n(t) = a_n e^{-(n\pi)^2 t} \quad n=1,2\ldots$

Into (*)

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-(n\pi)^2 t}\cos(n\pi x)$$

$a_0 = 1 \quad a_5 = 13$ and all others $a_n = 0$.

$$\boxed{u(x,t) = 1 + 13 e^{-(5\pi)^2 t}\cos(5\pi x)}$$

Q4.1

Solve 2point B.V.P.

$$\frac{d^2x}{dt^2} - x = 0$$

$x(0) = a \qquad x(1) = b$ B.C.

$r^2 - 1 = 0 \qquad r = \pm 1$

$$x(t) = c_1 \cosh t + c_2 \sinh t$$

$$a = x(0) = c_1 \cosh(0) + c_2 \sinh(0) = c_1$$

$$c_1 = a$$

$$b = x(1) = c_1 \cosh(1) + c_2 \sinh(1)$$
$$= a \cosh(1) + c_2 \sinh(1)$$

$$\frac{b - a\cosh(1)}{\sinh(1)} = c_2$$

$$\boxed{x(t) = a\cosh(t) + \frac{b - a\cosh(1)}{\sinh(1)} \sinh(t)}$$

Solve 2 point BVP

L 9 4.2

$$\frac{d^2x}{dt^2} + \pi^2 x = 0$$

$x(0) = a \qquad x(1) = b$ B.C.

$r^2 + \pi^2 = 0$

$r = \pm i\pi$

$x(t) = c_1 \cos(\pi t) + c_2 \sin(\pi t)$

$a = x(0) = c_1 \cos(\pi \cdot 0) + c_2 \sin(\pi \cdot 0) = c_1$

$b = x(1) = c_1 \cos(\pi \cdot 1) + c_2 \sin(\pi \cdot 1) = -c_1$

$c_1 = a$ but $c_1 = -b$??

what if $a \neq -b$??

Then there's no solution!

(9.5.1)

Derive formulae for the Fourier ~~coeffs~~.

$$f(x) = \sum_{n \geq 1}^{\infty} a_n \cos((n-1/2)\pi x)$$

Eigen functs
type IV B.C.

$$\{ \cos((n-1/2)\pi x) \}_{n \geq 1}^{\infty}$$

(a) Show this forms an orthogonal set.

$\underline{\underline{n \neq m}}$ $\quad \int_0^1 \cos((n-1/2)\pi x) \cos((m-1/2)\pi x)\,dx = 0$

$= 1, 2, \ldots$

$$= \frac{1}{2} \int_0^1 \left(\cos((n+m-1)\pi x) + \cos((n-m)\pi x) \right) dx$$

$$= \frac{1}{2} \left(\frac{\sin((n+m-1)\pi x)}{(n+m-1)\pi} + \frac{\sin((n-m)\pi x)}{(n-m)\pi} \right|_0^1$$

$$= \frac{1}{2} \left(\frac{\sin((n+m-1)\pi)}{(n+m-1)\pi} + \frac{\sin((n-m)\pi)}{(n-m)\pi} - \frac{\sin(0)}{n+m-1\,\pi} - \frac{\sin(0)}{(n-m)\pi} \right)$$

$$= \frac{1}{2}(0 + 0 - 0 - 0) = 0.$$

Also observe

$n = m = n$

$$\int_0^1 \cos^2((n-1/2)\pi x)\, dx = \frac{1}{2}\int_0^1 (1 + \cos(2(n-1/2)\pi x))$$

$$= \frac{1}{2} + \frac{1}{2}\left.\frac{\sin(2(n-1/2)\pi x)}{2(n-1/2)\pi}\right|_0^1 = 1/2$$

(b) get formulae for a_n's.

$$\int_0^1 f(x)\cos((m-1/2)\pi x) = \int_0^1 \sum_{n=1}^{\infty} a_n \cos((n-1/2)\pi x)\cos((m-1/2)\pi x)$$

$$= \sum_{n=1}^{\infty} a_n \underbrace{\left(\int_0^1 \cos((n-1/2)\pi x)\cos((m-1/2)\pi x)\, dx\right)}_{\boxed{=0 \ \forall n=1,2,\ldots \text{ except when } n=m}} !!$$

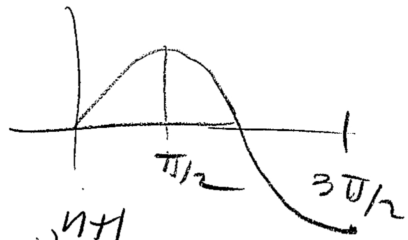
$$= a_m \int_0^1 \cos^2((m-1/2)\pi x)\, dx = \frac{1}{2} a_m$$

$$\Rightarrow \boxed{a_m = 2\int_0^1 f(x)\cos((m-1/2)\pi x)\, dx \quad \forall m = 1,2,\ldots}$$

(c) expand $f(x)=1$ into such a series (Q5.3

$$a_n = 2\int_0^1 1 \cos((n-1/2)\pi x)\,dx$$

$$= 2 \left. \frac{\sin((n-1/2)\pi x)}{(n-1/2)\pi} \right|_0^1$$



$$= 2 \frac{\sin((n-1/2)\pi)}{(n-1/2)\pi} = \frac{2(-1)^{n+1}}{(n-1/2)\pi}$$ cute huh?

$$\boxed{1 \sim \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{(n-1/2)\pi} \cos((n-1/2)\pi x)}$$

Solve the "inhomogeneous heat equation" (Q6i)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$\left.\begin{array}{l} u(0,t) = 280 \\ u(1,t) = 300 \end{array}\right\} \text{B.C.} \qquad u(x,0) = 280 \} \text{ I.C.}$$

not homogeneous

Find a special "steady state" soln.

$$0 = \frac{d^2 S}{dx^2}$$

$$\boxed{S = S(x) \quad \frac{\partial S}{\partial t} = 0}$$

$$S(0) = 280$$
$$S(1) = 300$$

Solve this 2 point B.V.P.

$$S(x) = C_1 + C_2 x$$

$$280 = S(0) = C_1$$

$$300 = S(1) = C_1 + C_2$$

$$S(x) = 280 + 20x.$$

12.6.2

$u(x,t) = S(x) + v(x,t)$

what eqn does v satisfy?

$$\frac{\partial}{\partial t}(S+v) = \frac{\partial^2}{\partial x^2}(S+v)$$

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}$$

$v(0,t) = u(0,t) - S(0) = 280 - 280 = 0$!

$v(1,t) = u(1,t) - S(1) = 300 - 300 = 0$!

B.C.s for v are

$\left.\begin{array}{l} v(0,t)=0 \\ v(1,t)=0 \end{array}\right\}$ homogeneous B.C.

$v(x,0) = u(x,0) - S(x) = 280 - (280 + 20x)$

$= -20x$

Gotta solve

$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}$, $\left.\begin{array}{l} v(0,t)=0 \\ v(1,t)=0 \end{array}\right\}$ B.C. $v(x,0) = -20x$

$$v(x,t) = \sum_{n=1}^{\infty} a_n e^{-(n\pi)^2 t} \sin(n\pi x)$$

$$a_n = 2\int_0^1 (-20y)\sin(n\pi y)\,dy$$

1 by P

Q6.3

$$a_n = -40\int_0^1 \underset{u}{y}\ \underset{dv}{\sin(n\pi y)\,dy}$$

$$= -40\left[-y\frac{\cos(n\pi y)}{n\pi}\Big|_0^1 + \cancel{\int_0^1 \frac{\cos(n\pi y)}{n\pi}\,dy}^{\,0} \right]$$

$$= -40\left[-\frac{\cos(n\pi)}{n\pi} \right] = \frac{40(-1)^n}{n\pi}$$

$$v(x,t) = \sum_{n=1}^{\infty} \frac{40(-1)^n}{n\pi} e^{-(n\pi)^2 t} \sin(n\pi x)$$

$$\boxed{\Rightarrow\quad u(x,t) = 280 + 20x + \sum_{n=1}^{\infty} \frac{40(-1)^n}{n\pi} e^{-(n\pi)^2 t} \sin(n\pi x)}$$

Note that

$$\lim_{t\to\infty} u(x,t) = 280 + 20x$$