you've obtained that (Q1.1)

$$u(x,t) = a_0 + b_0 t + \sum_{n=1}^{\infty} (a_n \cos(n\pi t) + b_n \sin(n\pi t)) \cos(n\pi x)$$

$u(x,0) = \cos(2\pi x) + 3\cos(3\pi x)$

$u_t(x,0) = 0$

$$\cos(2\pi x) + 3\cos(3\pi x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

$$0 = b_0 + \sum_{n=1}^{\infty} n\pi b_n \cos(n\pi x)$$

$\Rightarrow a_2 = 1 \quad a_3 = 3$ all other a_n's $= 0$

b_n's $= 0$ all n,

$$\boxed{u(x,t) = 1\cos(2\pi t)\cos(2\pi x) + 3\cos(3\pi t)\cos(3\pi x)}$$

Suppose by example we have instead $\boxed{Q\,11.2}$

$u(x,t) = a_0 + b_0 t + \sum_{n=1}^{\infty}$ same thing as before

$u(x,0) = 0$

$u_t(x,0) = 1 + 5\cos(6\pi x)$

$$0 = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

$$1 + 5\cos(6\pi x) = b_0 + \sum_{n=1}^{\infty} n\pi\, b_n \cos(n\pi x)$$

$\to a_n$'s $= 0 \quad \forall n,$

$b_0 = 1$

all other b_n's $= 0$.

$b_6 = \dfrac{5}{6\pi}$

$$\boxed{u(x,t) = 1 \cdot t + \frac{5}{6\pi}\sin(6\pi t)\cos(6\pi x)}$$

$$\frac{d^2 e}{dx^2} = \lambda e$$ $\boxed{\text{Solve This}}$ Q2.1

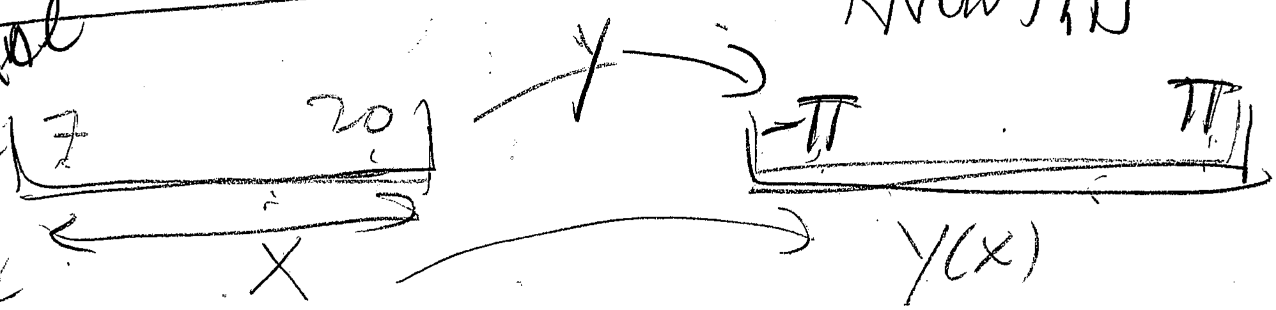
$$e(7) = e(20)$$

$$e'(7) = e'(20)$$

C.V. Will use change of variables

KNOW This

Determine change of variable



$$\boxed{y(x) = \frac{2\pi}{13}(x-7) - \pi}$$ ← This is prettier

$$y(x) = \alpha x + \beta \Rightarrow$$

$$y(x) = \frac{2\pi}{13}x - \frac{15\pi}{13}$$

$$-\pi = y(7) = 7\alpha + \beta$$

$$\pi = y(20) = 20\alpha + \beta$$

$$2\pi = 13\alpha \Rightarrow \alpha = \frac{2\pi}{13}$$

$$-\pi - 7\alpha = \beta$$

$$-\pi - \frac{2\pi}{13}\cdot 7 = \beta$$

$$-\pi - \frac{2\pi}{13}\cdot 7 = \beta \Rightarrow \beta = -\frac{15\pi}{13}$$

$\boxed{Q2,2}$

$$e(x) = \tilde{e}(y) \quad (**)$$

$$\frac{d}{dx} = \frac{dy}{dx}\frac{d}{dy} = \frac{2\pi}{13}\frac{d}{dy}$$

$$\frac{d^2}{dx^2} = \left(\frac{2\pi}{13}\right)^2 \frac{d^2}{dy^2}$$

$$\boxed{e'(x) = \frac{d}{dx}e(x) = \frac{dy}{dx}\frac{d}{dy}\tilde{e}(y) = \frac{2\pi}{13}\frac{d}{dy}\tilde{e}(y)}$$

$$\frac{d^2 e(x)}{dx^2} = \left(\frac{2\pi}{13}\right)^2 \frac{d^2}{dy^2}\tilde{e}(y) = \lambda \tilde{e}(y)$$

$$\frac{d^2\tilde{e}}{dy^2} = \frac{\lambda}{(2\pi/13)^2}\tilde{e} = \tilde{\lambda}\tilde{e}$$

where $\tilde{\lambda} = \lambda/(2\pi/13)^2 \quad (*)$

$$e(7) = \tilde{e}(-\pi) = e(20) = \tilde{e}(\pi)$$

B.C.

$$e'(7) = \frac{2\pi}{13}\tilde{e}'(-\pi) = e'(20) = \frac{2\pi}{13}\tilde{e}'(\pi)$$

B.C.

4 2.3

$$\frac{d^2\tilde{e}}{dy^2} = \tilde{\lambda}\tilde{e}$$

$$\tilde{e}(-\pi) = \tilde{e}(\pi)$$

$$\tilde{e}'(-\pi) = \tilde{e}'(\pi)$$

This is our old friend 2π period eval problem!

$$\tilde{\lambda}_0 = 0 \qquad \tilde{e}_0(y) = 1$$

$n = 1, 2, \ldots$

$$\tilde{\lambda}_n = -n^2 \qquad \tilde{e}_n(y) = \cos(ny)$$
$$\tilde{e}_n(y) = \sin(ny)$$

From (*) $\lambda = \left(\frac{2\pi}{13}\right)^2 \tilde{\lambda}$

From (**) $e(x) = \tilde{e}(y(x))$

$$\lambda_0 = 0 \qquad e_0(x) = 1$$

$n = 1, 2, \ldots$

$$\lambda_n = \left(\frac{2\pi}{13}\right)^2 (-n^2) \qquad e_n(x) = \cos\left(n\left(\frac{2\pi}{13}(x-7)-\pi\right)\right)$$
$$e_n(x) = \sin\left(n\left(\frac{2\pi}{13}(x-7)-\pi\right)\right)$$

To make even prettier $\boxed{Q2.4}$

$\lambda_0 = 0 \quad e_0(x) = 1$

$\lambda_n = -\left(\frac{2\pi}{13} n\right)^2 \quad e_n(x) = \cos\left(\frac{2\pi n}{13}(x-7)\right)$

$e_n(x) = \sin\left(\frac{2\pi n}{13}(x-7)\right)$

~~Prove it~~

In general

$$\frac{d^2 e}{dx^2} = \lambda e$$

$e(a) = e(b)$

$e'(a) = e'(b)$

$\lambda_0 = 0 \quad e_0(x) = 1$

$\lambda_n = -\left(\frac{2\pi}{b-a} n\right)^2 \quad e_n(x) = \cos\left(\frac{2\pi n}{b-a}(x-a)\right)$

$e_n(x) = \sin\left(\frac{2\pi n}{b-a}(x-b)\right)$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

Example Q3.1

where I always fool a bunch of students

$$\left.\begin{aligned} u(x,0) &= 0 \\ u_t(x,0) &= 1 \end{aligned}\right] \text{I.C.}$$

$$\boxed{u_x(0,t) = u_x(1,t) = 0 \quad \text{B.C.}}$$

The soln is $u(x,t) = t$.

$$\frac{d^2\alpha}{dt^2} = 0 \qquad e^{0t},\ te^{0t}$$

$$r^2 + 0 = 0$$

$$\alpha_0(t) = a_0 \cos(0t) + b_0 \boxed{\cancel{\sin(0t)}}$$ ← This is wrong

$$\alpha_0(t) = a_0 + b_0 t$$ ← This is correct

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{PDE}$$

$\boxed{Q4.}$

$$\left.\begin{aligned} u(-\pi,t) &= u(\pi,t) \\ u_x(-\pi,t) &= u(\pi,t) \end{aligned}\right\} \text{B.C.}$$

$$u(x,0) = |x| \quad \text{I.C.}$$

Solve

$$u(x,t) = \alpha_0(t) + \sum_{n=1}^{\infty} \left(\alpha_n(t)\cos(nx) + \beta_n(t)\sin(nx) \right)$$

Plug into PDE, to get

$$\frac{d\alpha_0}{dt} = 0 \cdot \alpha_0$$

$$\frac{d\alpha_n}{dt} = -n^2 \alpha_n$$

$$\frac{d\beta_n}{dt} = -n^2 \beta_n$$

I'll do this over

$\boxed{Q4.2}$

$$\frac{\partial}{\partial t}\left(\alpha_0 + \sum_{n=1} \alpha_n \cos nx + \beta_n \sin nx\right)$$

$$= \frac{\partial^2}{\partial x^2}\left(\alpha_0 + \sum_{n=1} \alpha_n \cos nx + \beta_n \sin nx\right)$$

$$LHS = \frac{\partial \alpha_0}{\partial t} \cdot 1 + \sum_{n=1} \frac{\partial \alpha_n}{\partial t} \cos nx + \frac{\partial \beta_n}{\partial t} \sin nx$$

$$RHS = 0 \cdot \alpha_0 \cdot 1 + \sum_{n=1}^{\infty} -n^2 \alpha_n \cos nx - n^2 \beta_n \sin nx$$

$$\frac{d\alpha_0}{dt} = 0 \cdot \alpha_0$$

$$\frac{d\alpha_n}{dt} = -n^2 \alpha_n$$

$$\frac{d\beta_n}{dt} = -n^2 \beta_n$$

Use d instead of ∂ because α & β are functions of t only.

Q 4.3

$$\alpha_0(t) = a_0 e^{0t}$$

$$\alpha_n(t) = a_n e^{-n^2 t} \qquad n = 1, 2, \ldots$$

$$\beta_n(t) = b_n e^{-n^2 t}$$

Soln to PDE is

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} e^{-n^2 t}(a_n \cos nx + b_n \sin nx)$$

What about the I.C. ??

~~u(x,0) = |x|~~

Expand $|x|$ into $\{1, \cos nx, \sin nx\}$

$$|x| = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |y|\,dy \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |y| \cos ny\,dy$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |y| \sin ny\,dy$$

~~Also~~ LQ 4.4

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|y|\,dy = \frac{\pi}{2}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}|y|\cos ny\,dy = \frac{2((-1)^n-1)}{n^2\pi}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}|y|\sin ny\,dy = 0$$

$$|x| = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2((-1)^n-1)}{n^2\pi}\cos nx + 0\sin nx$$

$$= u(x,0)$$

$$= a_0 + \sum_{n=1}^{\infty} e^{-n^2 0}(a_n\cos nx + b_n\sin nx)$$

$a_0 = \pi/2 \quad a_n = \frac{2((-1)^n-1)}{n^2\pi} \quad b_n = 0$

plug back into $u(x,t)$

Q 4.5

$$\boxed{u(x,t) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2((-1)^n - 1)}{n^2\pi} e^{-n^2 t} \cos nx}$$