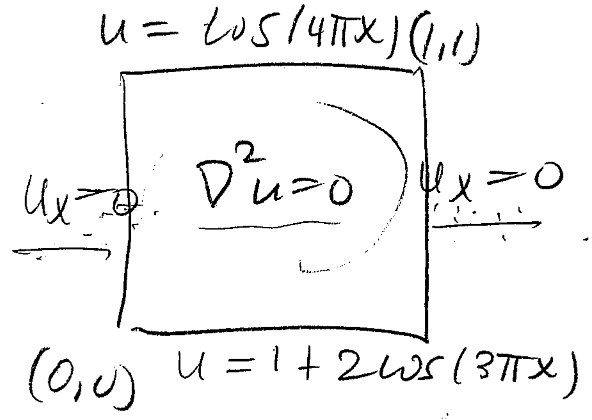
(Q1.1)



Seek $u(x,y)$ – the solution.

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Use an eigenfunction expansion

$$\begin{cases} \dfrac{d^2 e}{dx^2} = \lambda e & \\ e_x(0) = 0 \quad e_x(1) = 0 & \end{cases} \qquad \begin{aligned} e_n(x) &= \cos(n\pi x) \\ \lambda_n &= -(n\pi)^2 \\ n &= 0, 1, 2 \text{ ---} \end{aligned}$$

(*)

$$u(x,y) = \sum_{n=0}^{\infty} \underbrace{\alpha_n(y)}_{\text{to be determined}} \cos(n\pi x)$$

$$\nabla^2 u = \underbrace{\sum_{n=0}^{\infty} \alpha_n(y)\left(-(n\pi)^2\right)\cos(n\pi x)}_{\frac{\partial^2 u}{\partial x^2}} + \underbrace{\sum_{n=0}^{\infty} \frac{\partial^2 \alpha_n}{\partial y^2}\cos(n\pi x)}_{\frac{\partial^2 u}{\partial y^2}} = 0$$

(Q1.2

$$\underline{\nabla^2 u} = \sum_{n=0}^{\infty} \underbrace{\overbrace{\left(-(n\pi)^2 \alpha_n + \frac{\partial^2 \alpha_n}{\partial y^2} \right)}^{a_n}} \cos(n\pi x) \underset{\uparrow}{=} 0$$

for every x, y $\in [0,1] \times [0,1]$

$\Longleftrightarrow$ for each $n = 0, 1, 2 \ldots$

$$-(n\pi)^2 \alpha_n + \frac{d^2 \alpha_n}{d y^2} = 0$$

Have a seq of ODE's!!!

$$\frac{d^2 \alpha_n}{d y^2} - (n\pi)^2 \alpha_n = 0$$

$r^2 - (n\pi)^2 = 0$

$n \geq 1$ $\begin{cases} y'' - (n\pi)^2 y = 0 & r = \pm n\pi \\ y(y) = a_n \cosh(n\pi y) + b_n \sinh(n\pi y) \end{cases}$

$n = 0$ $\begin{cases} y'' = 0 \\ y(y) = a_0 + b_0 y \end{cases}$

i.e. $\alpha_n(y) = \begin{cases} a_0 + b_0 y & n = 0 \\ a_n \cosh(n\pi y) + b_n \sinh(n\pi y) & n \geq 1 \end{cases}$

$u(x,y)$ (plus into $*$ from prev page) (Q1.3)

(**) $= (a_0 + b_0 y) \cdot 1 + \sum_{n=1}^{\infty} \left(a_n \cosh(n\pi y) + b_n \sinh(n\pi y) \right) \cos(n\pi x)$

This satisfies B.C. on left right
(i.e. $u_x = 0$ at $x=0$ & $x=1$)

Also satisfies $\nabla^2 u = 0$ $\forall x,y \in [0,1] \times [0,1]$

Still need B.C. Bottom & Top
we'll determine the constants.

Use (**)

$$1 + 2\cos(3\pi x) = u(x,0)$$
$$= a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

$$\cos(4\pi x) = u(x,1)$$
$$= (a_0 + b_0) + \sum_{n=1}^{\infty} \begin{pmatrix} a_n \cosh(n\pi) \\ + b_n \sinh(n\pi) \end{pmatrix} \cos(n\pi x)$$

4 1, 4

$a_0 = 1$

$a_3 = 2$

a_n's $= 0$ otherwise

$a_0 + b_0 = 0$

$$a_n \cosh(n\pi) + b_n \sinh(n\pi) = \begin{cases} 0 & n \neq 4 \\ 1 & n = 4 \end{cases}$$

$b_0 = -1$

$$\underset{0}{\cancel{a_4}} \cosh(4\pi) + b_4 \sinh(4\pi) = 1$$

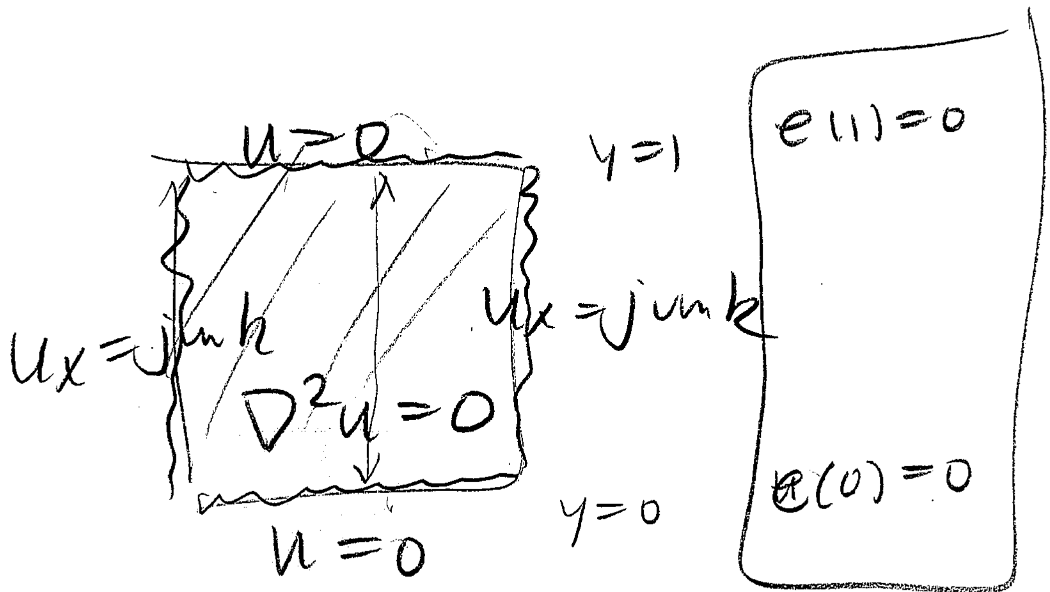
$$b_4 = \frac{1}{\sinh(4\pi)}$$

$$\overset{2}{\cancel{a_3}} \cosh(3\pi) + b_3 \sinh(3\pi) = 0$$

$$b_3 = -\frac{2 \cosh(3\pi)}{\sinh(3\pi)}$$

For other n's $\quad \overset{0}{\cancel{a_n}} \cosh(n\pi) + b_n \sinh(n\pi) = 0 \Rightarrow \boxed{b_n = 0}$

Q2.1



$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$u(x,y) = \sum_{n=1}^{\infty} \alpha_n(x) \sin(n\pi y)$$

$$\nabla^2 u = \overbrace{\sum_{n=1}^{\infty} \frac{\partial^2 \alpha_n}{\partial x^2} \sin(n\pi y)}^{\frac{\partial^2 u}{\partial x^2}} + \overbrace{\sum_{n=1}^{\infty} \alpha_n \left(-(n\pi)^2\right) \sin(n\pi y)}^{\frac{\partial^2 u}{\partial y^2}}$$

$$= \sum_{n=1}^{\infty} \left(\frac{\partial^2 \alpha_n}{\partial x^2} - (n\pi)^2 \alpha_n \right) \sin(n\pi y) \underset{\forall y}{=} 0$$

$$\iff \frac{d^2 \alpha_n}{dx^2} - (n\pi)^2 \alpha_n = 0 \quad n = 1, 2,$$

$\Phi 2,2$

$$X_n(x) = a_n \cosh(n\pi x) + b_n \sinh(n\pi x)$$

$(**)$

$$u(x,y) = \sum_{n=1}^{\infty} \left(a_n \cosh(n\pi x) + b_n \sinh(n\pi x) \right) \sin(n\pi y)$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \text{junk on left}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=1} = \text{junk on right}$$

keep going...

On homework

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \sin(\pi y)$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=1} = \sin(2\pi y)$$

Q2 i3

Use $(**)$

$$\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} n\pi \left(a_n \sinh(n\pi x) + b_n \cosh(n\pi x) \right) \sin(n\pi y)$$

$$\sin(\pi y) = \left.\frac{\partial u}{\partial x}\right|_{x=0} = \sum_{n=1}^{\infty} n\pi\, b_n \sin(n\pi y)$$

$$1 = \pi b_1 \Rightarrow \boxed{b_1 = \frac{1}{\pi}}$$

$$0 = n\pi b_n \quad n \neq 1$$

$b_n = 0$ otherwise

$$\sin(2\pi y) = \left.\frac{\partial u}{\partial x}\right|_{x=1} = \sum_{n=1}^{\infty} n\pi \left(a_n \sinh(n\pi) + b_n \cosh(n\pi) \right) \times \sin(n\pi y)$$

$n=2$

$$2\pi \left(a_2 \sinh(2\pi) + b_2 \cosh(2\pi) \right) = 1 \Rightarrow a_2 = \frac{1}{2\pi \sinh(2\pi)}$$

$$n\pi \left(a_n \sinh(n\pi) + b_n \cosh(n\pi) \right) = 0 \quad \forall n \neq 2$$

$$\pi \left(a_1 \sinh(\pi) + \frac{1}{\pi} \cosh(\pi) \right) = 0 \qquad a_1 = -\frac{\cosh(\pi)}{\pi \sinh(\pi)}$$

all other a_n's $= 0$

(Back to **)

Q2, 4

$$u(x,y) = \left(-\frac{\cosh(\pi)}{\pi \sinh(\pi)} \cosh(\pi x) + \frac{1}{\pi} \sinh(\pi x) \right) \sin(\pi y) + \left(\frac{1}{2\pi \sinh(2\pi)} \cosh(2\pi x) \right) \sin(2\pi y)$$