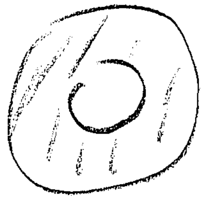
$\boxed{Q1.)}$

Solve $\nabla^2 u = 0$ $\quad 1 < r < 2$



$u(r,\theta)\big|_{r=1} = \cos\theta$

$u(r,\theta)\big|_{r=2} = \sin\theta$

$$u(r,\theta) = a_0 + \tilde{a}_0 \log(r) + \sum_{n=1}^{\infty} (a_n r^n + \tilde{a}_n r^{-n}) \cos n\theta + (b_n r^n + \tilde{b}_n r^{-n}) \sin n\theta$$

This is a $2\pi$ periodic (in $\theta$) soln to $\nabla^2 u = 0$

$$u(1,\theta) = \cos\theta = \left(a_0 + \tilde{a}_0 \log(1)\right) + \sum_{n\geq 1} (a_n + \tilde{a}_n) \cos n\theta + (b_n + \tilde{b}_n) \sin n\theta$$

$$\underline{a_0 + \tilde{a}_0 \ln(1) = 0}$$

$$\underline{(a_1 + \tilde{a}_1) = 1}$$

$\forall n \geq 2$ $\quad (a_n + \tilde{a}_n) = 0$

$\forall n \geq 1$ $\quad \underline{(b_n + \tilde{b}_n)} = 0$

$$u(2,\theta) = \sin\theta = \left(a_0 + \tilde{a}_0 \log(2)\right) + \sum_{n=1}^{\infty} (a_n 2^n + \tilde{a}_n 2^{-n}) \cos n\theta + (b_n 2^n + \tilde{b}_n 2^{-n}) \sin n\theta$$

Q1, 2

$$a_0 + \tilde{a}_0 \log(2) = 0$$

$\forall n \geq 1$ $(a_n 2^n + \tilde{a}_n 2^{-n}) = 0$

$$(b_1 2^1 + \tilde{b}_1 2^{-1}) = 1$$

$\forall n \geq 2$ $(b_n 2^n + \tilde{b}_n 2^{-n}) = 0$

Gotta solve all of these simultaneously

$$a_0 + \tilde{a}_0 \ln(1) = 0$$
$$a_0 + \tilde{a}_0 \ln(2) = 0$$

2 eqns in 2? soln is $a_0 = 0$ $\tilde{a}_0 = 0$

$$a_1 + \tilde{a}_1 = 1$$
$$a_1 \cdot 2 + \tilde{a}_1 2^{-1} = 0$$

2 eqns in 2? $a_1 = -1/3$ $\tilde{a}_1 = 4/3$

$$b_1 \cdot 2 + \tilde{b}_1 2^{-1} = 1.$$
$$b_1 + \tilde{b}_1 = 0$$

2 eqns in 2? $b_1 = +\frac{2}{3}$ $\tilde{b}_1 = -\frac{2}{3}$

$\forall n \geq 2$ determine

$$a_n = \tilde{a}_n = 0$$
$$b_n = \tilde{b}_n = 0$$

$$u(r,\theta) = \left(-\tfrac{1}{3} r^1 + \tfrac{4}{3} r^{-1}\right)\cos\theta + \left(\tfrac{2}{3} r^1 - \tfrac{2}{3} r^{-1}\right)\sin\theta$$

$u(1,\theta) = \cos\theta + 0$ $u(2,\theta) = \sin\theta$ ✓