Q1.

$$\hat{f}(w) = \frac{1}{2\pi}\int_{\mathbb{R}} f(x)\, e^{+iwx}\, dx$$ + on the forward trans.

$$f(x) = \int_{\mathbb{R}} \hat{f}(w)\, e^{-iwx}\, dw$$ − on the inverse transform.

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

Try $u(x,t) = \int_{\mathbb{R}} \alpha(w,t)\, e^{-iwx}\, dw$

to be determined.

$$\int_{\mathbb{R}} \frac{d^2\alpha}{\partial t^2} e^{-iwx} dw = \int_{\mathbb{R}} (-iw)^2 \alpha\, e^{-iwx} dw$$

2nd order linear const coeff.

$$\Rightarrow \frac{d^2\alpha}{dt^2} + w^2\alpha = 0 \quad \forall w$$

$r^2 + w^2 = 0$

$$\Rightarrow \alpha(w,t) = a(w) e^{-iwt} + b(w) e^{iwt}$$

$\forall w \neq 0$.

$$u(x,t) = \int_{\mathbb{R}} \left(a(w) e^{-iwt} + b(w) e^{iwt} \right) e^{-iwx} \, dw$$

(*)
$$= \int_{\mathbb{R}} \left(a(w) e^{-iw(x+t)} + b(w) e^{-iw(x-t)} \right) dw$$

Suppose we have initial conds

$$u(x,0) = 0 = \int_{\mathbb{R}} 0 \, e^{-iwx} \, dw$$

$$u_t(x,0) = g(x)$$
$$= \int_{\mathbb{R}} \hat{g}(w) e^{-iwx} \, dw$$

Set $t=0$ in (*) $\quad \left[\, 0 = a(w) + b(w) \right.$

$\frac{\partial}{\partial t}$, set $t=0$ in (*) $\quad \left[\, \hat{g}(w) = -iw\, a(w) + iw\, b(w) \right.$

Solve 2 eqns in 2?

$$\Rightarrow a(w) = -\frac{1}{2iw} \hat{g}(w), \quad b(w) = \frac{1}{2iw} \hat{g}(w)$$

$$u(x,t) = \frac{1}{2}\left[\int_{\mathbb{R}} -\frac{\hat{g}(w)}{iw} e^{-iw(x+t)}\,dw \right.$$

(*)

$$\left. + \int_{\mathbb{R}} \frac{\hat{g}(w)}{iw} e^{-iw(x-t)}\,dw\right]$$

$$A(\theta) = \int_{\mathbb{R}} \frac{\hat{g}(w)}{-iw} e^{-iw\theta}\,dw$$

$$\frac{d A(\theta)}{d\theta} = \int_{\mathbb{R}} \hat{g}(w) e^{-iw\theta}\,dw = g(\theta)$$

$$\underset{\text{FTC}}{\Rightarrow} A(\theta) = \underset{\text{number}}{\underset{\uparrow}{A_0}} + \int_0^\theta g(s)\,ds$$

Use (*) $\quad \theta = x+t$

$$u(x,t) = \frac{1}{2}\left[\left(A_0 + \int_0^{x+t} g(s)\,ds\right) - \left(A_0 + \int_0^{x-t} g(s)\,ds\right)\right]$$

$$= \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds \quad //$$

4.2.1

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}$$

in $-\infty < t < \infty$

$u(x,0) = f(x)$

$$u(x,t) = \int_{\mathbb{R}} \alpha(w,t) e^{-iwx}\, dw$$

(*)

$$\int_{\mathbb{R}} \frac{\partial \alpha}{\partial t} e^{-iwx}\, dw = \int_{\mathbb{R}} -iw\alpha e^{-i\alpha x}\, dw$$

$$\Rightarrow \frac{d\alpha}{dt} + iw\alpha = 0$$

$$\underline{\alpha(w,t)} = a(w) e^{-iwt} \quad \forall w$$

use (*)

$$u(x,t) = \int_{\mathbb{R}} a(w) e^{-iwt} e^{-iwx}\, dw = \int_{\mathbb{R}} a(w) e^{-iw(x+t)}\, dw$$

(**)

$$u(x,0) = f(x)$$

$$= \int_{\mathbb{R}} \hat{f}(w) e^{-iwx}\, dw \overset{\text{use } (**)}{=} \underbrace{\int_{\mathbb{R}} a(w) e^{-iwx}\, dw}_{u(x,0)}$$

$$\Rightarrow a(w) = \hat{f}(w)$$

So

$$u(x,t) = \int_{\mathbb{R}} \hat{f}(w) e^{-iw(x+t)}\, dw$$

But $$\underbrace{\int_{\mathbb{R}} \hat{f}(w) e^{-iw\theta}\, dw}_{\text{inverse F.T. of the F.T.}} = f(\theta)$$

$$\Rightarrow \boxed{u(x,t) = f(x+t)}$$