$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} \\ u(x,0) = f(x) \\ u(x,t) = \dfrac{1}{\sqrt{4\pi t}} \displaystyle\int_{-\infty}^{\infty} f(s) e^{-\frac{(x-s)^2}{4t}} ds \end{cases}$$

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nu \dfrac{\partial^2 u}{\partial x^2} \qquad \nu > 0 \\ u(x,0) = f(x) \end{cases}$$

$$u(x,t) = W(x,\tau)$$

$$\nu t = \tau$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} \frac{\partial \tau}{\partial t} = \nu \frac{\partial}{\partial \tau}$$

$$\cancel{\nu} \frac{\partial W}{\partial \tau} = \cancel{\nu} \frac{\partial^2 W}{\partial x^2} \qquad W(x,0) = u(x,0) = f(x)$$

$$\frac{\partial w}{\partial \tau} = \frac{\partial^2 w}{\partial x^2}$$

$$w(x,0) = f(x)$$

$$w(x,\tau) = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{\infty} f(s)\, e^{-\frac{(x-s)^2}{4\tau}}\, ds$$

$u(x,t)$

$$\tau = \nu t$$

$$\frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} f(s)\, e^{-\frac{(x-s)^2}{4\nu t}}\, ds$$

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2} \quad (*) \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

$$u(x,t) = \frac{1}{2}\left(f(x-t) + f(x+t)\right) + \frac{1}{2}\int_{x-t}^{x+t} g(s)\, ds$$

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

$$u(x,t) = ?????$$

$$ct = \tau$$

$$\frac{\partial}{\partial t} = \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} = c\frac{\partial}{\partial \tau}$$

$$\frac{\partial^2}{\partial t^2} = c^2\frac{\partial^2}{\partial \tau^2}$$

$$w(x, \tau) = u(x, t)$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 w}{\partial \tau^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 w}{\partial x^2}$$

$$\frac{\partial^2 w}{\partial \tau^2} = \frac{\partial^2 w}{\partial x^2}$$

$$w(x, 0) = f(x)$$

$$w_\tau(x, 0) = \frac{1}{c}g(x)$$

$$w_\tau(x, 0) = \frac{1}{c}u_t(x, 0) = \frac{1}{c}g(x)$$

Use (*)

$$W(x,\tau)$$

$$= \frac{1}{2}\left(f(x-\tau) + f(x+\tau)\right)$$

$$+ \frac{1}{2}\int_{x-\tau}^{x+\tau} \frac{1}{c}\, g(s)\, ds$$

$$u(x,t) = W(x, ct)$$

$$= \frac{1}{2}\left(f(x-ct) + f(x+ct)\right)$$

$$+ \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\, ds$$