In class we worked out that the solution to transport-diffusion equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x)$$

is

(*) $$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(s)\, e^{-\frac{(x-t-s)^2}{4t}}\, ds$$

Use this to determine the solution of

$$\boxed{\begin{aligned} &\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2} \qquad (\varepsilon > 0) \\ &u(x,0) = f(x) \end{aligned}}$$

Change variables: $x = Ly$, $t = T\tau$

$$u(x,t) = v(y,\tau)$$

$$\frac{1}{T}\frac{\partial v}{\partial \tau} + \frac{1}{L}\frac{\partial v}{\partial y} = \frac{\varepsilon}{L^2}\frac{\partial^2 v}{\partial y^2}$$

$$v(y,0) = u(x,0) = u(Ly,0) = f(Ly)$$

So $$\frac{\partial v}{\partial \tau} + \frac{T}{L}\frac{\partial v}{\partial y} = \frac{\varepsilon T}{L^2}\frac{\partial^2 v}{\partial y^2}$$

To make this into "parent" equation we need $\frac{T}{L} = 1$ $\frac{\varepsilon T}{L^2} = 1$

Solution is $T = L = \varepsilon$.

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial y} = \frac{\partial^2 v}{\partial y^2}$$

$$v(y,0) = f(\varepsilon y)$$

Use (*) to get

$$v(y,\tau) = \frac{1}{\sqrt{4\pi\tau}}\int_{-\infty}^{\infty} f(\varepsilon s)\, e^{\frac{-(y-\tau-s)^2}{4\tau}}\, ds$$

But $u(x,t) = v(y,\tau) = v\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right)$

$$= \frac{1}{\sqrt{4\pi t/\varepsilon}}\int_{-\infty}^{\infty} f(\varepsilon s)\, e^{\frac{-(x/\varepsilon - t/\varepsilon - s)^2}{4t/\varepsilon}}\, ds$$

over

Change variables in integral, $\varepsilon s = z$
and simplify

$$u(x,t) = \frac{1}{\sqrt{4\pi t/\varepsilon}} \int_{-\infty}^{\infty} f(z)\, e^{-\frac{(x/\varepsilon - t/\varepsilon - z/\varepsilon)^2}{4t/\varepsilon}} \frac{dz}{\varepsilon}$$

$$u(x,t) = \frac{1}{\sqrt{4\pi t\varepsilon}} \int_{-\infty}^{\infty} f(z)\, e^{-\frac{(x-t-z)^2}{4t\varepsilon}}\, dz$$

Q1.1

$$u(x,t) = \int_{-\infty}^{\infty} \hat{u}(w,t) e^{-iwx} \, dw$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 \hat{u}}{\partial t^2} = -w^2 \hat{u}$$

$$\boxed{r^2 + w^2 = 0}$$

$$r = \pm iw$$

ODE solve it

$$\hat{u}(w,t) = a(w) e^{iwt} + b(w) e^{-iwt}$$

$$u(x,t) = \int_{-\infty}^{\infty} \left(a(w) e^{iwt} + b(w) e^{-iwt}\right) e^{-iwx} \, dw$$

$$u(x,0) = f(x) = \int_{-\infty}^{\infty} \hat{f}(w) e^{-iwx} \, dw$$

$$= \int_{-\infty}^{\infty} \left(a(w) + b(w)\right) e^{-iwx} \, dw$$

$$\Leftrightarrow \quad a(w) + b(w) = \hat{f}(w)$$

9l.2

$$u_t(x,0) = 0 = \int_{-\infty}^{\infty} 0\, e^{-i\omega x}\, d\omega$$

$$= \int_{-\infty}^{\infty} \left(i\omega a(\omega) - i\omega b(\omega) \right) e^{-i\omega x}\, d\omega$$

$$\Leftrightarrow \boxed{0 = \underset{\omega \neq 0}{i\omega a(\omega) - i\omega b(\omega)}}$$

$\underline{\omega \neq 0}$

$$a + b = \hat{f}$$
$$a - b = 0$$

$$\begin{aligned} 2a &= \hat{f} \\ 2b &= \hat{f} \end{aligned} \Rightarrow \begin{aligned} a &= \hat{f}/2 \\ b &= \hat{f}/2 \end{aligned}$$

$$u(x,t) = \frac{1}{2} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-i\omega(x-t)}\, d\omega$$

$$+ \frac{1}{2} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-i\omega(x+t)}\, d\omega$$

$\underbrace{\text{inv F.T. of } \hat{f} \to f}$ Q1.3

Since

$$f(\theta) = \int_{-\infty}^{\infty} \hat{f}(w) e^{-iw\theta}\, dw$$

for any $\theta \in \mathbb{R}$

$$\int_{-\infty}^{\infty} \hat{f}(w) e^{-iw(x-t)}\, dw = f(x-t)$$

$$\int_{-\infty}^{\infty} \hat{f}(w) e^{-iw(x+t)}\, dw = f(x+t)$$

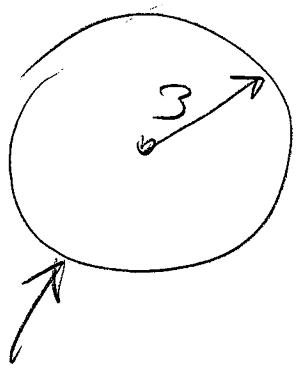
So I get

$$\boxed{u(x,t) = \frac{1}{2}\left(f(x-t) + f(x+t)\right)}$$

Done

$\nabla^2 u = 0$ in polar coords. Q 2.1

$$u(r,\theta) = a_0 + \tilde{a}_0 \ln(r) + \sum_{n=1}^{\infty} \left(a_n r^n + \tilde{a}_n r^{-n}\right) \cos(n\theta) + \sum_{n=1}^{\infty} \left(b_n r^n + \tilde{b}_n r^{-n}\right) \sin(n\theta)$$



$r < 3$
origin is inside domain
must not include singular
blow up terms
terms.

$$u(3,\theta) = 1 + 2\cos(2\theta) + 4\sin(3\theta)$$

$$1 + 2\cos(2\theta) + 4\sin(3\theta) = a_0 + \sum_{n=1}^{\infty} a_n 3^n \cos(n\theta) + \sum_{n=1}^{\infty} b_n 3^n \sin(n\theta)$$

$a_0 = 1 \quad a_2 3^2 = 2 \quad b_3 3^3 = 4$

$a_0 = 1 \quad a_2 = 2/9 \quad b_3 = 4/27$

all others must be zero.

$$\boxed{u(r,\theta) = 1 + \frac{2}{9} r^2 \cos(2\theta) + \frac{4}{27} r^3 \sin(3\theta)}$$

Q 2.2

$\frac{1}{2} < r < 3$

$u(3,\theta) = \cos(2\theta)$

$u(1/2,\theta) = 2$

@ $r = 1/2$

$$\underline{2} = \underline{a_0 + \tilde{a}_0 \ln(1/2)} + \sum_{n=1}^{\infty} \left(a_n \left(\tfrac{1}{2}\right)^n + \tilde{a}_n \left(\tfrac{1}{2}\right)^{-n}\right)\cos(n\theta)$$
$$+ \sum_{n=1}^{\infty} \left(b_n \left(\tfrac{1}{2}\right)^n + \tilde{b}_n \left(\tfrac{1}{2}\right)^{-n}\right)\sin(n\theta)$$

(*) $2 = a_0 + \tilde{a}_0 \ln(1/2)$

(**) $a_n \left(\frac{1}{2}\right)^n + \tilde{a}_n \left(\frac{1}{2}\right)^{-n} = 0$

(***) $b_n \left(\frac{1}{2}\right)^n + \tilde{b}_n \left(\frac{1}{2}\right)^{-n} = 0$

$n \geq 1$

$\phi 2,3$

$$\cos(2\theta) = a_0 + \tilde{a}_0 \ln(3) + \sum_{n=1}^{\infty} (a_n 3^n + \tilde{a}_n 3^{-n}) \cos(n\theta) + \sum_{n=1}^{\infty} (b_n 3^n + \tilde{b}_n 3^{-n}) \sin(n\theta)$$

$(*)$ $0 = a_0 + \tilde{a}_0 \ln(3)$

$(**)$ $1 = (a_2 3^2 + \tilde{a}_2 3^{-2})$

$$(***)\begin{cases} 0 = (a_n 3^n + \tilde{a}_n 3^{-n}) & \forall n \geq 1 \ n \neq 2 \\ 0 = (b_n 3^n + \tilde{b}_n 3^{-n}) & \forall n \geq 1 \end{cases}$$

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used $(*)$

$$\begin{cases} a_0 + \tilde{a}_0 \ln(1/2) = 2 \\ a_0 + \tilde{a}_0 \ln(3) = 0 \end{cases}$$

$$\tilde{a}_0 (\ln(1/2) - \ln(3)) = 2$$

$$\tilde{a}_0 = \frac{2}{\ln(1/6)} \quad \Big| \quad a_0 = -\frac{2\ln(3)}{\ln(1/6)}$$

§2.4

use (**)

$$3^2 a_2 + 3^{-2}\tilde{a}_2 = 1$$

$$\left(\tfrac{1}{2}\right)^2 a_2 + \left(\tfrac{1}{2}\right)^{-2}\tilde{a}_2 = 0$$

$$a_2 = \frac{\det\begin{pmatrix} 1 & 1/9 \\ 0 & 4 \end{pmatrix}}{\det\begin{pmatrix} 9 & 1/9 \\ 1/4 & 4 \end{pmatrix}} = \frac{4}{36 - \frac{1}{36}}$$

$$\tilde{a}_2 = \frac{\det\begin{pmatrix} 9 & 1 \\ 1/4 & 0 \end{pmatrix}}{\det\begin{pmatrix} 9 & 1/9 \\ 1/4 & 4 \end{pmatrix}} = \frac{-1/4}{36 - \frac{1}{36}}$$

On the other ones

use (**)

$$\left(\tfrac{1}{2}\right)^n a_n + \left(\tfrac{1}{2}\right)^{-n}\tilde{a}_n = 0$$

$$3^n a_n + 3^{-n}\tilde{a}_n = 0$$

$$\Longleftrightarrow a_n = \tilde{a}_n = 0 \qquad \forall n \neq 2, \ n \neq 0$$

$$(1/2)^n b_n + \left(\tfrac{1}{2}\right)^{-n}\tilde{b}_n = 0$$

$$(3^n)\, b_n + 3^{-n}\tilde{b}_n = 0 \qquad \Longleftrightarrow \quad b_n = \tilde{b}_n = 0 \quad \forall n$$

Q2,5

$$\boxed{\begin{aligned} u(r,\theta) &= -\frac{2\ln(3)}{\ln(1/6)} + \frac{2}{\ln(1/6)}\ln(r) \\ &+ \left(\frac{4}{36-\frac{1}{36}} r^{2} - \frac{1/4}{36-1/36} r^{-2}\right)\cos(2\theta) \end{aligned}}$$

$+ 0$ Check

plug in $r = 1/2$

$$u(1/2,\theta) = -\frac{2\ln(3)}{\ln(1/6)} + \frac{2\ln(1/2)}{\ln(1/6)}$$

$$= \frac{2(-\ln(2)-\ln(3))}{-\ln(6)} = \frac{-2\ln(6)}{-\ln(6)} = 2$$

plug in $r=3$

$$0 + \left(\frac{4\cdot 9}{36-\frac{1}{36}} - \frac{1/4 \cdot 1/9}{36-\frac{1}{36}}\right)\cos(2\theta)$$

$$= \frac{36 - 1/36}{36-1/36}\cos(2\theta) = \cos(2\theta)$$