Math 3363 Sanders Spring 2008

Some Homework Questions for Exam 1

1. Determine whether or not the given operators, $\mathcal{L}(u)$, are linear. ($u = u(x)$)

(a) $\mathcal{L}(u) = \dfrac{du}{dx} + xu.$ (c) $\mathcal{L}(u) = \dfrac{du}{dx} + x.$

(b) $\mathcal{L}(u) = \dfrac{du}{dx} + \displaystyle\int_0^x u(y)\,dy.$ (d) $\mathcal{L}(u) = \dfrac{du}{dx} + |u|.$

Throughout, the following boundary condition types are denoted by

Type I:	$u(0) = 0,$	$u(1) = 0.$
Type II:	$u_x(0) = 0,$	$u_x(1) = 0.$
Type III:	$u(0) = 0,$	$u_x(1) = 0.$
Type IV:	$u_x(0) = 0,$	$u(1) = 0.$
Type V:	$u(-\pi) = u(\pi),$	$u_x(-\pi) = u_x(\pi).$

2. Calculate all eigenvalues and eigenfunctions for the operator $\mathcal{L}(u) = \dfrac{d^2u}{dx^2}$ subject to each of the five types of boundary conditions given above. (You may only assume that the eigenvalues are real; nothing more. They also must be correctly enumerated.)

3. By directly calculating integrals, establish that the eigenfunction obtained in exercise 2 are orthogonal with respect to the inner product

$$(f, g) = \int_0^1 f(x)g(x)\,dx \quad \text{for types I–IV}, \qquad (f, g) = \int_{-\pi}^{\pi} f(x)g(x)\,dx \quad \text{for type V.}$$

4. Use your results already derived in exercise 2 to determine all the eigenvalues and eigenfunctions for $\mathcal{L}(u) = \dfrac{d^2u}{dx^2}$ on the interval $(0, L)$ for types I–IV boundary conditions, and $(-L, L)$ for type V.

5. Find all eigenvalues and eigenfunctions for the operator $\mathcal{L}(u) = \dfrac{d^2u}{dx^2} - 2\dfrac{du}{dx}$ defined on the interval $(0, 1)$ subject to type I and type II boundary conditions given above. (You may only assume that the eigenvalues are real; nothing more. They also must be correctly enumerated.)

Answers:

Type I: $\lambda_n = -1 - (n\pi)^2,\ u_n(x) = e^x \sin(n\pi x),$ for $n = 1, 2, 3, \ldots$.

Type II: $\lambda_0 = 0,\ u_0(x) = 1,$ as well as

$\lambda_n = -1 - (n\pi)^2,\ u_n(x) = e^x(n\pi\cos(n\pi x) - \sin(n\pi x)),$ for $n = 1, 2, \ldots$.

6. Expand the function $f(x) = 1$ into a Fourier series based on each of the five sets of eigenfunctions derived in exercise 2. Do the same for $f(x) = x$, and $f(x) = x^2$.

7. Expand each of the following functions into a Fourier series based on the type I only eigenfunctions derived in exercise 2.

(a) $f(x) = \begin{cases} 0 & x < 1/2 \\ 1 & x \geq 1/2. \end{cases}$ (b) $f(x) = 1 - 2|x - 1/2|$.

8. Solve the heat equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ subject to

(a) $\begin{cases} \text{type I boundary conditions, and} \\ u(x,0) = 2\sin(3\pi x) + 4\sin(5\pi x). \end{cases}$ (b) $\begin{cases} \text{type II boundary conditions, and} \\ u(x,0) = 2 + 3\cos(4\pi x) \end{cases}$.

9. Solve $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ with type I boundary conditions and initial condition $u(x,0) = 1$.

10. Solve the wave equation $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$ subject to

(a) $\begin{cases} \text{type I boundary conditions, and} \\ u(x,0) = 2\sin(3\pi x) + 4\sin(5\pi x) \\ u_t(x,0) = 0. \end{cases}$ (b) $\begin{cases} \text{type II boundary conditions, and} \\ u(x,0) = 0 \\ u_t(x,0) = 1. \end{cases}$