The Five Eigenvalue Problems

When

$$\frac{d^2e}{dx^2} = \lambda e, \text{ where } \lambda \text{ is the eigenvalue and } e \text{ its associated eigenfunction,}$$

the given boundary conditions give rise to the eigenvalue/eigenfunction pairs.

$$e(0) = 0,\ e(1) = 0 \quad \Rightarrow \quad \begin{array}{l} \text{for } n = 1, 2, 3, \ldots \\ \lambda_n = -(n\pi)^2 \\ e_n(x) = \sin(n\pi x). \end{array}$$

$$e'(0) = 0,\ e'(1) = 0 \quad \Rightarrow \quad \begin{array}{l} \lambda_0 = 0,\ e_0(x) = 1, \text{ and for} \\ n = 1, 2, 3, \ldots \\ \lambda_n = -(n\pi)^2 \\ e_n(x) = \cos(n\pi x). \end{array}$$

$$e(0) = 0,\ e'(1) = 0 \quad \Rightarrow \quad \begin{array}{l} \text{for } n = 1, 2, 3, \ldots \\ \lambda_n = -((n - 1/2)\pi)^2 \\ e_n(x) = \sin((n - 1/2)\pi x). \end{array}$$

$$e'(0) = 0,\ e(1) = 0 \quad \Rightarrow \quad \begin{array}{l} \text{for } n = 1, 2, 3, \ldots \\ \lambda_n = -((n - 1/2)\pi)^2 \\ e_n(x) = \cos((n - 1/2)\pi x). \end{array}$$

$$\begin{array}{l} e(-\pi) = e(\pi), \\ e'(-\pi) = e'(\pi) \end{array} \quad \Rightarrow \quad \begin{array}{l} \lambda_0 = 0,\ e_0(x) = 1, \text{ and for} \\ n = 1, 2, 3, \ldots \\ \lambda_n = -n^2 \\ e_n(x) = \cos(nx),\ e_n(x) = \sin(nx) \end{array}$$

Derive these and memorize for the rest of the semester.