10 The First Gap Theorem

Theorem 10.1 (The First Gap Theorem) For $N < 2n - 1$, any map $F \in Prop_2(\mathbb{B}^n, \mathbb{B}^N)$ is equivalent to the linear map $(z, 0, w)$.

This theorem is a result by many mathematicians over 20 years.

In 1979, S. Webster proved [W79] that any mapping in $Prop_3(\mathbb{B}^n, \mathbb{B}^{n+1})$ with $n \geq 3$ must be equivalent to a linear map $(z, 0, w)$.

In 1982, J. Faran [Fa82] proved that there are exactly four maps in $Prop_3(\mathbb{B}^2, \mathbb{B}^3)$, up to equivalence class.

Next year, A. Cima and T.J. Suffridge [CS83] improved the above results of Webster and Faran by replacing “$Prop_3$” with “$Prop_2$”. In the same paper [CS83], A. Cima and T. J. Suffridge conjectured that any mapping in $Prop_2(\mathbb{B}^n, \mathbb{B}^N)$ with $n \geq 3$ and $N \leq 2n - 2$ should be equivalent to the linear map $(z, 0, w)$.

In 1986, Faran [Fa86] proved the Cima-Suffridge’s conjecture under the assumption that $F$ is holomorphic in a neighborhood of $\mathbb{B}^n$.

In the same year, F. Forstnerič [Fo86] proved $Prop_{N-n+1}(\mathbb{B}^n, \mathbb{B}^N) = Rat(\mathbb{B}^n, \mathbb{B}^N)$ and later Cima and Suffridge [CS90] shown that any mapping in $Rat(\mathbb{B}^n, \mathbb{B}^N)$ must be holomorphic on the boundary. As a consequence, the First Gap Theorem is proved for any $F \in Prop_{N-n+1}(\mathbb{B}^n, \mathbb{B}^N)$ with $N < 2n - 1$. 
In 1999 X. Huang [Hu99] proved that any mapping in $Prop_2(\mathbb{B}^n, \mathbb{B}^N)$ with $N \leq 2n - 2$ is equivalent to the linear map $(z, 0, w)$.

**Outline of the Proof for the First Gap Theorem:**

**Step 1.** if $N < 2n - 1$, it implies that its geometric rank $\kappa_0 = 0$.

- *(analytic proof)* Use Uniqueness theorem (see Corollary 20.1 and Theorem 20.2 below).
- *(geometric proof)* Use the formula

$$N \geq n + \frac{(2n - \kappa_0 - 1)\kappa_0}{2}$$

for any $F \in Prop_2(\mathbb{B}^n, \mathbb{B}^N)$ with geometric rank $\kappa_0$. In fact, if $N < 2n - 1$, the above inequality forces $\kappa_0 = 0$.

**Step 2.** Show: $\kappa_0 = 0 \iff F$ is a linear fractional map.

- *(analytic proof)* The first order PDE argument (see Theorem 19.1 below).
- *(geometric proof)* $\kappa_0 = 0 \iff$ the CR second fundamental form $II_M = 0 \iff F$ is a linear fractional map. □

We need to explain the following:

1. What is the geometric rank $\kappa_0$ of a map $F$? (see (102) below, or [HJ01])
2. Why $N \geq n + \frac{(2n - \kappa_0 - 1)\kappa_0}{2}$? (see Corollary 32.2, or [H03])
3. Why $\kappa_0$ if and only if $II_M = 0$? (see Corollary 44.3, [JY09][HJ09])
4. Why $II_M = 0$ if and only if $F$ is a linear fractional map (see Theorem 39.1, [JY09]).
11 Passing from $\partial B^n$ to $\partial H^n$

Recall the Heisenberg hypersurface

$$\partial H^n := \{(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} : \text{Im}(w) = |z|^2\}$$

and the Cayley transformation

$$\rho_n : H^n \to B^n, \quad \rho_n(z, w) = \left(\frac{2z}{1-iw}, \frac{1+iw}{1-iw}\right).$$

We can define the space $\text{Prop}(H^n, H^N)$, $\text{Prop}_k(H^n, H^N)$ and $\text{Rat}(H^n, H^N)$. We can identify a map $F \in \text{Prop}_k(B^n, B^N)$ or $\text{Rat}(B^n, B^N)$ with $\rho_N^{-1} \circ F \circ \rho_n$ in the space $\text{Prop}_k(H^n, H^N)$ or $\text{Rat}(H^n, H^N)$, respectively.

We say that $F$ and $G \in \text{Prop}(H^n, H^N)$ are equivalent if there are automorphisms $\sigma \in \text{Aut}(H^n)$ and $\tau \in \text{Aut}(H^N)$ such that $F = \tau \circ G \circ \sigma$.

$$B^n \xrightarrow{F} B^N \xleftarrow{\rho_n} H^n \xrightarrow{\rho_N^{-1} \circ F \circ \rho_n} N^N.$$  

12 Differential Operators on $\partial H^n$

The vector fields $\{L_1, ..., L_{n-1}\}$, where $L_j := 2iz_j \frac{\partial}{\partial w} + \frac{\partial}{\partial z_j}$, form a global basis for the complex tangent bundle $\mathbb{C}T^{1,0}\partial H^n$ over $\partial H^n$, and their conjugates $\{\overline{L}_1, ..., \overline{L}_{n-1}\}$, called CR vector fields, form a global basis for the complex tangent bundle $\mathbb{C}T^{0,1}\partial H^n$ over $\partial H^n$. Recall that for $z_j = x_j + iy_j$ and for $w = u + iv$, we have

$$\frac{\partial}{\partial z_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right), \quad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right).$$

and

$$\frac{\partial}{\partial w} = \frac{1}{2} \left( \frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right), \quad \frac{\partial}{\partial \overline{w}} = \frac{1}{2} \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right).$$

There is a real vector field which is transversal to $\mathbb{C}T^{(1,0)}\partial H^n + \mathbb{C}T^{(0,1)}\partial H^n$

$$T = \frac{\partial}{\partial \text{Re}(w)} = \frac{\partial}{\partial u} = \frac{\partial}{\partial w} + \frac{\partial}{\partial \overline{w}}.$$  

(29)
which is the Reeb vector field.

The vector fields \( \{ L_1, \ldots, L_{n-1}, \overline{L}_1, \ldots, \overline{L}_{n-1}, T \} \) forms a basis of \( \mathbb{C}T\partial\mathbb{H}_n \).

**Lemma 12.1**

(i) \( TL_j = L_j T, \overline{T}L_j = \overline{L}_j T, \) and \( L_j L_k = L_k L_j \) for all \( 1 \leq j, k \leq n - 1 \).

(ii) For any continuous CR function \( h \) over an open subset \( M_1 \subset \partial\mathbb{H}^n \), \( T h \) is a CR distribution over \( M_1 \). For any \( 1 \leq j, k \leq n - 1 \), \( \overline{T}L_k(\overline{L}_j h) = -[L_j, L_k]h = 2i\delta_{kj}T h \).

(iii) Let \( h \) be a \( C^2 \) CR function over \( \partial\mathbb{H}^n \) and \( \chi \) a \( C^1 \) function over \( \partial\mathbb{H}^n \). Then for any integer \( k > 0 \), we have

\[
\overline{L}_k(L^2_k(h)\chi) = 4iL_k(T(h))\chi + L^2_k(h)L_k(\chi),
\]

in the sense of distribution.

(iv) For any \( k, l, j \) and any \( C^2 \) CR function \( h \), we have

\[
\overline{L}_kL_lL_jh = 2i\delta_{kl}TL_jh + 2i\delta_{kj}TL_lh
\]

in the sense of distribution. In particular, we have

\[
\overline{L}_kL_lL_jh = \begin{cases} 
0, & \text{when } k \neq l \text{ and } k \neq j; \\
2iT(L_lh), & \text{when } k = j \neq l; \\
2iT(L_jh), & \text{when } k = l \neq j; \\
4iT(L_kh), & \text{when } k = j = l.
\end{cases}
\]

**Proof of Lemma 12.1**

(i) For any differentiable function \( f(z, \overline{z}, w, \overline{w}) \),

\[
T(L_jf) = \left( \frac{\partial}{\partial w} + \frac{\partial}{\partial \overline{w}} \right) \left( \frac{\partial f}{\partial z_j} + 2i\overline{z}_j \frac{\partial f}{\partial w} \right) = \frac{\partial^2 f}{\partial w \partial z_j} + 2i\overline{z}_j \frac{\partial^2 f}{\partial w^2} + \frac{\partial^2 f}{\partial \overline{w} \partial z_j} + 2i\overline{z}_j \frac{\partial^2 f}{\partial \overline{w} \overline{w}}
\]

\[
L_j(Tf) = \left( \frac{\partial}{\partial \overline{z}_j} + 2i\overline{z}_j \frac{\partial}{\partial \overline{w}} \right) \left( \frac{\partial f}{\partial \overline{w}} + \frac{\partial f}{\partial \overline{w}} \right) = \frac{\partial^2 f}{\partial \overline{w} \partial z_j} + 2i\overline{z}_j \frac{\partial^2 f}{\partial \overline{w}^2} + \frac{\partial^2 f}{\partial w \partial \overline{z}_j} + 2i\overline{z}_j \frac{\partial^2 f}{\partial w \overline{w}}.
\]

Then \( TL_j = L_j T \) and hence \( \overline{T}L_j = \overline{L}_j T \). Similarly, \( L_j L_k = L_k L_j, \forall 1 \leq j, k \leq n - 1. \)
(ii) The first statement follows from (i): \( T h \) is CR because \( \overline{L_j} Th = T \overline{L_j} h = 0 \). The second statement follows from the following calculation:

\[
[L_j, \overline{L_k}] = \left( \frac{\partial}{\partial z_j} + 2i\overline{z_j} \frac{\partial}{\partial w} \right) \left( \frac{\partial}{\partial \overline{z_k}} - 2i \overline{z_k} \frac{\partial}{\partial \overline{w}} \right) - \left( \frac{\partial}{\partial \overline{z_k}} - 2i \overline{z_k} \frac{\partial}{\partial \overline{w}} \right) \left( \frac{\partial}{\partial z_j} + 2i z_j \frac{\partial}{\partial w} \right)
\]

\[
= -2i \delta_{jk} \frac{\partial}{\partial \overline{w}} - 2i \delta_{j \overline{k}} \frac{\partial}{\partial w} = -2i \delta_{kj} T.
\]

(iii) It is sufficient to prove (iii) for any holomorphic polynomial \( h \) by a lemma below. By (ii), we know that \( T h \) is CR and that \( \overline{L_k} L_k h = 2iT h \). This follows the second identity. To prove the first identity, it is sufficient to prove

\[
\overline{L_k} L_k^2 h = 4iL_k Th, \quad \forall \ C^2 \text{ CR function } h.
\] (30)

In fact, \( \overline{L_k} L_k^2 h \) equals to

\[
([\overline{L_k}, L_k] + L_k \overline{L_k}) L_k h = 2iT L_k h + L_k ([\overline{L_k}, L_k] + L_k \overline{L_k}) h = 2iT L_k h + 2iL_k Th + 0 = 4iT L_k h.
\]

(iv) It is sufficient to prove (iv) for any holomorphic polynomial \( h \) as above. Consider

\[
\overline{L_k} L_\ell L_j h = ([\overline{L_k}, L_\ell] + L_\ell \overline{L_k}) L_j h = 2i \delta_{k \ell} T L_j h + L_\ell ([\overline{L_k}, L_j] + L_j \overline{L_k}) h = 2i \delta_{k \ell} T L_j h + L_\ell 2i \delta_{k j} T h + 0 = 2i \delta_{k \ell} T L_j h + 2i \delta_{k j} T L_\ell h
\]

\[
= \begin{cases} 
0, & \text{if } k \neq j, \ell \neq k \\
2iT L_\ell h, & \text{if } k = j, \ell \neq j \\
2iT L_j h, & \text{if } k = \ell \neq j \\
4iT L_k h, & \text{if } j = k = \ell.
\end{cases}
\]

by using the similar computation. □

Let \( h \) be a \( C^v \)-smooth function and then \( D_1(h) \) is a \( C^0 \)-smooth function for any differential operator \( D_1 \) of degree \( v \). Let \( D_2 \) be another differential operator. In general \( D_2 D_1(h) \) does not make sense. However if \( D_2 D_1(h) \) can be written as \( D_3(h) \) where \( D_3 \) is of degree \( v \). Then \( D_2 D_1(h) \) is still a \( C^0 \) function. This fact is presented by a lemma below. As an example, \( \overline{L_j} L_1 h = 2i \delta_{ji} T h \). It can also been seen in Lemma 12.1 (ii) and (iii).
Lemma 12.2 Let $h$ be a $C^v$-smooth CR map from a neighborhood of $M$ in $\partial \mathbb{H}_n$ into $\mathbb{C}^N$. Let $D_1(h) = H(p, \overline{p}, L^a L^b T^c(h))_{|a|+|b|+|c|\leq v}$ with $H$ holomorphic in its argument where $p \in \partial \mathbb{H}_n$. Let $D_2 = L^a_1 L^b_1 T^c_1$ be a differential operator along $M$. Suppose that there is a certain holomorphic function $H_0$ in its argument such that for each polynomial map $h^*$ from $\mathbb{C}^n$ into $\mathbb{C}^N$,

$$D_2(D_1(h^*)) = H_0(p, \overline{p}, L^a_2 L^b_2 T^c_2(h^*))_{|a_2|+|b_2|+|c_2|\leq v}$$

Then the distribution $D_2(D_1(h))$, acting on $C^\infty_0(M)$, coincides with the continuous function $D_3(h) := H_0(p, \overline{p}, L^a_2 L^b_2 T^c_2(h))_{|a_2|+|b_2|+|c_2|\leq v}$.

Proof of Lemma 12.2: It is an immediate application of the Baouendi-Treves approximation theorem. Here we outline the proof. There is a sequence of holomorphic polynomial maps $\{h_m\}_{m=1}^\infty$ which converges to $h$ in the $C^v$-norms over $\overline{M}$. Hence $D_1(h_m) \to D_1(h)$ in the $C^0$-norm over $\overline{M}$, and $D_2(D_1(h_m)) \to D_2(D_1(h))$ in the sense of distribution. By the assumption, $D_2(D_1(h_m))$ converges also to $H_0(p, \overline{p}, L^a_2 L^b_2 T^c_2(G))_{|a|+|b|+|c|\leq v}$ in the $C^0$-norm over $\overline{M}$. $\square$