Lecture 5. Zeno’s Four Paradoxes of Motion

Science of infinity    In Lecture 4, we mentioned that a conflict arose from the discovery of irrationals. The Greeks’ rejection of irrational numbers was essentially part of a general rejection of the infinite process. In fact, until the late 19th century most mathematicians were reluctant to accept infinity. In his last observations on mathematics \(^1\), Hermann Weyl wrote: “Mathematics has been called the science of the infinite. Indeed, the mathematician invents finite constructions by which questions are decided that by their very nature refer to the infinite. That is his glory.”

Besides the conflict of irrational numbers, there were other conflicts about infinite process: the famous Zeno’s paradoxes of motion around 450 B.C.

Zeno of Citium

Figure 5.1  Zeno of Citium

Zeno’s Paradoxes of motion    Zeno (490 B.C. - 430 B.C.) was a Greek philosopher of southern Italy and a member of the Eleatic School founded by Parmenides \(^2\). We know


\(^2\)Parmenides of Elea, early 5th century B.C., was an ancient Greek philosopher born in Elea, a Greek city on the southern coast of Italy. He was the founder of the Eleatic school of philosophy. For his picture, see Figure 2.3.
little about Zeno other than Plato’s assertion that he went to Athens to meet with Socrates when he was nearly 40 years old. Zeno was a Pythagorean. By a legend, he was killed by a tyrant of Elea whom he had plotted to depose.

The Eleatics were a school of pre-Socratic philosophers at Elea, a Greek colony in Campania, Italy. The school debated the possibility of motion and other such fundamental questions.

Zeno was best known for his paradoxes. He proposed four paradoxes in an effort to challenge the accepted notions of space and time that he encountered in various philosophical circles. His paradoxes confounded mathematicians for centuries.

\[ A \rightarrow D \rightarrow C \rightarrow B \]  

\textbf{Figure 5.2.}

- **Dichotomy paradox:**

  There is no motion because that which is moved must arrive at the middle (of its course) before it arrives at the end. (Aristotle, Physics, Book VI, Ch.9)

In other words (see Figure 5.2), if one wants to traverse \( AB \), one must first arrive at \( C \); to arrive at \( C \) one must first arrive \( D \); and so forth. In other words, on the assumption that space is infinitely divisible and therefore that a finite length contains an infinite number of points, it is impossible to cover even a finite length in a finite time.

- **Achilles and the tortoise paradox:** A fleet-of-foot Achilles \(^3\) is unable to catch a plodding tortoise which has been given a head start, since during the time it takes Achilles to catch up to a given position, the tortoise has moved forward some distance. But this is obviously not true since Achilles will clearly pass the tortoise! The resolution is similar to that of the dichotomy paradox.

\(^3\)In Greek mythology, Achilles was a Greek hero of the Trojan War, the central character and the greatest warrior of Homer’s \textit{Iliad}. Achilles also was also a hero against Troy.
• **Arrow paradox:** An arrow in flight has an instantaneous position at a given instant of time. At that instant, the arrow must occupy a particular position in space, i.e., the arrow is at rest; at every instant, it is at rest. Since the arrow must always occupy such a position on its trajectory which is equal to its length, the arrow must be always at rest. Therefore flying arrow is motionless.

![Diagram of arrows and points](image)

• **Stadium paradox:** It is about bodies moving in opposite directions with equal speed, and Zeno concluded that twice the speed is the same as half the speed. In other words, assume that there are three sets of identical objects, the A’s at rest, the B’s moving...
to the right pass the A’s, and the C’s moving to the left with equal velocity. Suppose that B’s have moved one place to the right and the C’s one place to the left, so that $B_1$, which was originally under $A_4$ is now under $A_5$, while $C_1$, originally under $A_5$ is now under $A_4$. Zeno supposes that the objects are indivisible elements of space (like atoms) and that they move to their new positions in an indivisible unit of time (also like atoms). As B’s moved to A by one unit, B’s is moved towards C by two units, by indivisibility of the units, $\frac{1}{2}(\text{unit})=1(\text{unit})$. Thus, half a time is equal to the whole time.

4See the textbook: p.46, the 3rd edition; p. 57, the 2nd edition.

Figure 5.4 The ancient Agora (looking southwest). Beginning at around 301 B.C., Zeno taught philosophy at the Stoa Poikile (the painted porch), from which his philosophy got its name. Unlike the other schools of philosophy, such as the Epicureans, Zeno chose to teach his philosophy in a public space, which was a colonnade overlooking the central gathering place of Athens, the Agora.

All of the four paradoxes are essentially the same, each involves an understanding of the “infinite process.” Controversy regarding these paradoxes has lasted throughout history. At the time, Aristotle argued that Zeno’s fallacy consisted in supporting that things that move with the same speed past a moving object and past a fixed object take the same time. Neither Zeno’s argument nor Aristotle’s answer is clear.

From today’s point of view, Zeno’s Paradoxes can been explained by the infinite series theory in Calculus. For example, Zeno’s paradox of the dichotomy concerns the decompo-
tion of the number 1 into the infinite series

\[ 1 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \ldots \]

If we accept the concept of convergence, the paradox is settled.

It took a long time for people to understand the infinity process (series theory, and the concepts of continuity, limit and convergence). The discovering of irrational numbers was one event in mathematical history, and Zeno’s Four Paradoxes was another.

Galileo wrote in his *Dialogues Concerning Two New Sciences* in 1638: “Infinities and indivisibles transcend our finite understanding, the former on account of their magnitude, the latter because of their smallness; Imagine what they are when combined.”

**Archimedes’ exhaustion method** Before 212 B.C., Archimedes had developed a method to derive a finite answer for the sum of infinitely many terms that get progressively smaller. The method allows for a construction of solutions stating that (under suitable conditions), if these distances are decreasing at a sufficiently rapid rate, the travel time is finite (bounded by a certain amount).

By Archimedes’ exhaustion method, the Greeks were quite close to the solution of understanding the infinite process, but this did not occur. Even in the earlier years of calculus, the nature of infinity and the infinite process confused mathematicians over a long period.

**Zeno’s thought and the modern physics** Zeno thought that a line could be divided into an infinite number of points, without limit. However in the modern quantum physics, it was believed that the smallest distance may be the Plank distance of $10^{-33}$ centimeters, where the fabric of space-time becomes foamy and bubbly. But the M-theory gives a new twist: the physics within the Planck length is identical to the physics outside the Planck length so that even within the “smallest distance of string theory, an entire universe can exist. Zeno’s thought is right.

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5 Translated by Henry Crew and Alfonso De Salvio, New York: Dover, 1914.
6 For more information about Archimedes, see Lecture 12.