# COMPUTING EXT FOR GRAPH ALGEBRAS (UNABRIDGED)

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Abstract. For a row-finite graph $G$ with no sinks and in which every loop has an exit, we construct an isomorphism between $\mathrm{Ext}(C^*(G))$ and $\mathrm{coker}(A - I)$, where $A$ is the vertex matrix of $G$. If $c$ is the class in $\mathrm{Ext}(C^*(G))$ associated to a graph obtained by attaching a sink to $G$, then this isomorphism maps $c$ to the class of a vector that describes how the sink was added. We conclude with an application in which we use this isomorphism to produce an example of a row-finite transitive graph with no sinks whose associated $C^*$-algebra is not semiprojective.

## 1. Introduction

The Cuntz-Krieger algebras $\mathcal{O}_A$ are $C^*$-algebras that are generated by a collection of partial isometries satisfying relations described by a finite matrix $A$ with entries in $\{0, 1\}$ and no zero rows. In [5] Cuntz and Krieger computed Ext for these $C^*$-algebras, showing that $\mathrm{Ext}(\mathcal{O}_A)$ is isomorphic to $\mathrm{coker}(A - I)$, where $A : \mathbb{Z}^n \to \mathbb{Z}^n$.

In 1982 Watatani noted that one can view $\mathcal{O}_A$ as the $C^*$-algebra of a finite directed graph $G$ with no sinks and whose vertex adjacency matrix is $A$ [19]. However, it was not until the late 1990's that analogues of these $C^*$-algebras were considered for possibly infinite graphs that are allowed to contain sinks [9, 10]. Since that time there has been a flurry of activity in studying these graph algebras.

Graph algebras have proven to be important for many reasons. To begin with, they include a fairly wide class of $C^*$-algebras. In addition to generalizing the Cuntz-Krieger algebras, graph algebras include many other interesting classes of $C^*$-algebras such as AF-algebras and Kirchberg-Phillips algebras with torsion-free $K_1$-group. However, despite the fact that graph algebras include a wide class of $C^*$-algebras, their basic structure is fairly well understood and their invariants are readily computable. In fact, results about Cuntz-Krieger algebras can often be extended to graph algebras with only minor modifications. One reason graph algebras have attracted the interest of many people is that the graph provides a convenient tool for visualization. Not only does the graph determine the defining relations for the generators of the $C^*$-algebra, but also many important properties of the $C^*$-algebra may be translated into graph properties that can easily be read off from the graph.

In this paper we extend Cuntz and Krieger's computation of $\mathrm{Ext}\,\mathcal{O}_A$ to graph algebras. Specifically, we prove the following.

**Theorem.** *Let $G$ be a row-finite graph with no sinks and in which every loop has an exit, and let $C^*(G)$ be the $C^*$-algebra associated to $G$. Then there exists an*

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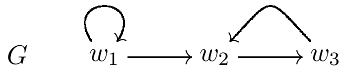
*isomorphism*

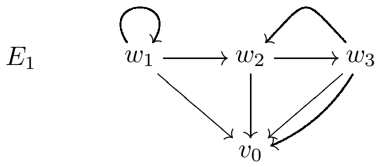
$$\omega : \mathrm{Ext}(C^*(G)) \to \mathrm{coker}(A_G - I)$$

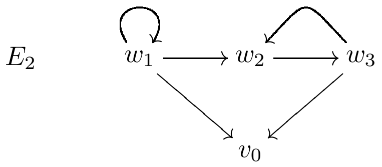
*where* $A_G$ *is the vertex matrix of* $G$ *and* $A_G : \prod_{G^0} \mathbb{Z} \to \prod_{G^0} \mathbb{Z}$.

In addition to showing that $\mathrm{Ext}(C^*(G)) \cong \mathrm{coker}(A_G - I)$, the isomorphism $\omega$ is important because its value on certain extensions can be easily calculated. If $E$ is an essential 1-sink extension of $G$ as described in [13], then $C^*(E)$ will be an extension of $C^*(G)$ by $\mathcal{K}$ and thus determines an element in $\mathrm{Ext}(C^*(G))$. Roughly speaking, a 1-sink extension of $G$ may be thought of as a graph formed by attaching a sink $v_0$ to $G$, and this 1-sink extension is said to be essential if every vertex of $G$ can reach this sink. For example, if $G$ is the graph

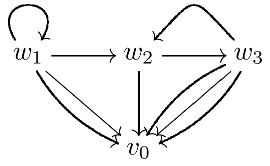


then two examples of essential 1-sink extensions are the following graphs $E_1$ and $E_2$.





For each 1-sink extension there is a vector, called the Wojciech vector, that describes how the sink is added to $G$ [13]. In the above two examples the Wojciech vector is the vector whose $v^{\text{th}}$ entry is equal to the number of edges from $v$ to the sink. This vector is $\begin{pmatrix}1\\1\\2\end{pmatrix}$ for $E_1$ and $\begin{pmatrix}1\\0\\1\end{pmatrix}$ for $E_2$. It turns out that if $E$ is a 1-sink extension of $G$, then the value that $\omega$ assigns to the element of $\mathrm{Ext}(C^*(G))$ associated to $E$ is equal to the class of the Wojciech vector of $E$ in $\mathrm{coker}(A_G - I)$. Furthermore, since $\omega$ is additive we have a nice way of describing addition of elements in $\mathrm{Ext}(C^*(G))$ associated to essential 1-sink extensions. For example, if $E_1$ and $E_2$ are as above, then the sum of their associated elements in $\mathrm{Ext}(C^*(G))$ is the element in $\mathrm{Ext}(C^*(G))$ associated to the 1-sink extension



whose Wojciech vector is $\begin{pmatrix}2\\1\\3\end{pmatrix} = \begin{pmatrix}1\\1\\2\end{pmatrix} + \begin{pmatrix}1\\0\\1\end{pmatrix}$. Thus we have a way of visualizing certain elements of $\mathrm{Ext}(C^*(G))$ as well as a way to visualize their sums. We show in §6 that if $G$ is a finite graph, then every element of $\mathrm{Ext}(C^*(G))$ is an element associated to an essential 1-sink extension of $G$. We also show that this is not necessarily the case for infinite graphs.

In addition to providing an easily visualized description of $\mathrm{Ext}(C^*(G))$, we also show that the isomorphism $\omega$ can be used to ascertain information about the semiprojectivity of a graph algebra. Blackadar has shown that the Cuntz-Krieger algebras are semiprojective [4], and Szymański has proven that $C^*$-algebras of transitive graphs with finitely many vertices are semiprojective [16]. Although not all graph algebras are semiprojective (for instance, it follows from [4, Theorem 3.1] that $\mathcal{K}$ is not semiprojective), it is natural to wonder if the $C^*$-algebras of transitive graphs will always be semiprojective. In §7 we answer this question in the negative. We use the isomorphism $\omega$ to produce an example of a row-finite transitive graph whose $C^*$-algebra is not semiprojective.

This paper is organized as follows. We begin in §2 with the definition of Ext and some preliminary facts regarding it. In §3 we provide a description of Ext due to Cuntz and Krieger and prove its equivalence to the usual definition. We believe that this result is of independent interest since it gives a more tractable way of determining the equivalence classes that make up Ext and because it applies to arbitrary $C^*$-algebras and not just graph algebras. After some graph algebra preliminaries in §4, we continue in §5 by creating a map $d : \mathrm{Ext}(C^*(G)) \to \mathrm{coker}(B_G - I)$, where $B_G$ is the edge matrix of $G$. In §6 we define the map $\omega : \mathrm{Ext}(C^*(G)) \to \mathrm{coker}(A_G - I)$, where $A_G$ is the vertex matrix of $G$. We also prove that $\omega$ is an isomorphism and compute the value it assigns to elements of $\mathrm{Ext}(C^*(G))$ associated to essential 1-sink extensions. We conclude in §7 by providing an example of a row-finite transitive graph whose $C^*$-algebra is not semiprojective.

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## 2. Ext Preliminaries

Throughout we shall let $\mathcal{H}$ denote a separable infinite-dimensional Hilbert space, $\mathcal{K}$ denote the compact operators on $\mathcal{H}$, $\mathcal{B}$ denote the bounded operators on $\mathcal{H}$ , and $\mathcal{Q} := \mathcal{B}/\mathcal{K}$ denote the associated Calkin algebra. We shall also let $i : \mathcal{K} \to \mathcal{B}$ denote the inclusion map and $\pi : \mathcal{B} \to \mathcal{Q}$ denote the projection map.

In this section we review a few definitions and establish notation that is used in defining Ext. For a more complete treatment of Ext we refer the reader to [3] and [8], or for a less comprehensive but more introductory treatment we suggest [18].

We will assume that the reader is familiar with Busby invariants of extensions. In particular, if $A$ is a $C^*$-algebra, then an *extension* of $A$ (by the compact operators) is a homomorphism $\tau : A \to \mathcal{Q}$. An extension is said to be *essential* if it is a monomorphism.

*Definition* 2.1. An extension $\tau : A \to \mathcal{Q}$ is said to be *degenerate* if there exists a homomorphism $\eta : A \to \mathcal{B}$ such that $\pi \circ \eta = \tau$. In other words, $\tau$ can be lifted to a (possibly degenerate) representation $\eta$.

We warn the reader that the terminology used above is not standard. Many authors refer to such extensions as trivial rather than degenerate. However, we have chosen to follow the convention established in [8].

*Definition* 2.2. Two extensions $\tau_1, \tau_2 : A \to \mathcal{Q}$ are *strongly equivalent* if there exists a unitary $u \in \mathcal{B}$ such that $\tau_1 = \mathrm{Ad}(\pi(u)) \circ \tau_2$. In this case we write $\tau_1 \approx \tau_2$.

We now define a binary operation on strong equivalence classes of extensions. Choose any isomorphism $\Theta : M_2(\mathcal{K}) \to \mathcal{K}$. This induces an isomorphism $\tilde{\Theta} : M_2(\mathcal{Q}) \to \mathcal{Q}$. For two extensions $\tau_1$ and $\tau_2$ we define $\tau_1 + \tau_2$ to be the homomorphism given by $(\tau_1 + \tau_2)(a) = \tilde{\Theta}((\tau_1 \oplus \tau_2)(a))$, where $(\tau_1 \oplus \tau_2)(a) = \begin{pmatrix} \tau_1(a) & 0 \\ 0 & \tau_2(a) \end{pmatrix}$. This gives a well-defined associative operation on strong equivalence classes of extensions. With this operation the strong equivalence classes form a commutative semigroup [8, Lemma 2.3] and the degenerate extensions form a subsemigroup. We define $\operatorname{Ext}(A)$ to be what Blackadar calls the quotient by this subsemigroup. Formally, this entails the following.

*Definition* 2.3. Two extensions $\tau_1$ and $\tau_2$ are said to be *stably equivalent* if there are degenerate extensions $\lambda_1$ and $\lambda_2$ such that $\tau_1 + \lambda_1 \approx \tau_2 + \lambda_2$. In this case we write $\tau_1 \sim \tau_2$. $\operatorname{Ext}(A)$ is defined to be the commutative semigroup consisting of stable equivalence classes of extensions.

*Definition* 2.4. Two extensions $\tau_1, \tau_2 : A \to \mathcal{Q}$ are *weakly equivalent* if there exists a unitary $u \in \mathcal{Q}$ such that $\tau_1 = \operatorname{Ad}(u) \circ \tau_2$. In this case we write $\tau_1 \approx_w \tau_2$.

One can prove that the operation + is well-defined on weak equivalence classes of extensions and that with this operation the weak equivalence classes are a commutative semigroup. Once again the degenerate extensions form a subsemigroup and we may form the quotient. Formally,

*Definition* 2.5. Two extensions $\tau_1$ and $\tau_2$ are said to be *weakly stably equivalent* if there are degenerate extensions $\lambda_1$ and $\lambda_2$ such that $\tau_1 + \lambda_1 \approx_w \tau_2 + \lambda_2$. In this case we write $\tau_1 \sim_w \tau_2$. $\operatorname{Ext}_w(A)$ is defined to be the commutative semigroup consisting of weakly stable equivalence classes of extensions.

It turns out that stable equivalence classes coincide with weak stable equivalence classes; that is $\operatorname{Ext}(A) = \operatorname{Ext}_w(A)$ [3, Proposition 15.6.4]. Furthermore the essential extensions play an important role.

*Definition* 2.6. We define $\operatorname{Ext}_w^e(A)$ to be the quotient of weak equivalence classes of essential extensions by the subsemigroup of degenerate essential extensions. Thus $\tau_1$ and $\tau_2$ give the same class in $\operatorname{Ext}_w^e(A)$ if there are essential degenerate extensions $\lambda_1$ and $\lambda_2$ such that $\tau_1 + \lambda_1 \approx_w \tau_2 + \lambda_2$.

**Proposition 2.7.** *If there exists an essential degenerate extension of $A$ by $\mathcal{K}$, then the natural map of $\operatorname{Ext}_w^e(A)$ into $\operatorname{Ext}_w(A)$ is an isomorphism.*

The above is a corollary of [3, Proposition 15.6.5]. This proposition shows us that in most cases we may identify $\operatorname{Ext}_w^e(A)$ with $\operatorname{Ext}_w(A) = \operatorname{Ext}(A)$. Thus we may restrict our attention to the weak stable equivalence classes of essential extensions. This fact will be important in the following section when we discuss Cuntz and Krieger's description of $\operatorname{Ext}(A)$.

## 3. Cuntz and Krieger's description of Ext

In [5] Cuntz and Krieger computed $\operatorname{Ext} \mathcal{O}_A$ using a slightly nonstandard definition of Ext. We will want to make use of this description of Ext, so in this section we give an expanded version of it and prove that in general it is equivalent to the usual definition.

*Definition* 3.1. We say that two Busby invariants $\tau_1$ and $\tau_2$ are *CK-equivalent* if there exists a partial isometry $v \in \mathcal{Q}$ such that

$$\tau_1 = \operatorname{Ad}(v) \circ \tau_2 \quad \text{and} \quad \tau_2 = \operatorname{Ad}(v^*) \circ \tau_1. \tag{3.1}$$

*Remark* 3.2. Note that CK-equivalence is clearly reflexive and symmetric. However, because $v$ is a partial isometry it is not obvious whether it is also transitive. In the following lemma we show that two essential extensions are CK-equivalent if and only if they are weakly stably equivalent. Hence CK-equivalence is transitive for essential extensions. It is unclear to the author at this time whether CK-equivalence is transitive in general.

**Lemma 3.3.** *Suppose that* $\tau_1$ *and* $\tau_2$ *are the Busby invariants of two essential extensions of* $A$ *by* $\mathcal{K}$. *Then* $\tau_1$ *equals* $\tau_2$ *in* $\operatorname{Ext}_w^e(A)$ *if and only if* $\tau_1$ *and* $\tau_2$ *are CK-equivalent.*

Before giving the proof we need a couple of observations.

**Lemma 3.4.** *Suppose that* $\lambda_1$ *and* $\lambda_2$ *are essential degenerate extensions of* $A$ *by* $\mathcal{K}$. *Then there is a partial isometry* $v \in \mathcal{Q}$ *such that*

$$\lambda_1 = \operatorname{Ad}(v) \circ \lambda_2 \qquad\qquad \lambda_2 = \operatorname{Ad}(v^*) \circ \lambda_1.$$

*Thus all essential degenerate extensions are CK-equivalent.*

*Proof.* Let $\lambda_i = \pi \circ \hat{\lambda}_i$ for possibly degenerate representations $\hat{\lambda}_i : A \to \mathcal{B}(\mathcal{H})$. Since the $\hat{\lambda}_i$ may be degenerate representations, we can't apply Voiculescu's Theorem [17] directly. However, recall that

$$\mathcal{H}_i := \overline{\operatorname{span}}\{\hat{\lambda}_i(a)h : a \in A \text{ and } h \in \mathcal{H}\}$$

is an invariant subspace for $\hat{\lambda}_i$ and is called the *essential subspace* of $\hat{\lambda}_i$. The restriction ess $\hat{\lambda}_i$ of $\hat{\lambda}_i$ to $\mathcal{H}_i$ is nondegenerate and is called the essential part of $\hat{\lambda}_i$. Note that $\hat{\lambda}_i = \operatorname{ess}\, \hat{\lambda}_i \oplus 0$. Since each $\lambda_i$ is faithful by assumption, ess $\hat{\lambda}_i$ is injective and ess $\hat{\lambda}_i(A) \cap \mathcal{K}(\mathcal{H}_i) = 0$. Now Voiculescu's Theorem [17] implies that there is a unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$(\operatorname{ess}\, \hat{\lambda}_2)(a) - U(\operatorname{ess}\, \hat{\lambda}_1)(a)U^*$$

is compact for all $a \in A$. It follows that there is a partial isometry $V \in \mathcal{B}(\mathcal{H})$ such that both

$$\hat{\lambda}_2(a) - V\hat{\lambda}_1(a)V^* \quad \text{ and } \quad \hat{\lambda}_1(a) - V^*\hat{\lambda}_2(a)V$$

are compact for all $a \in A$. The result follows. □

**Lemma 3.5.** *Let* $\tau_1$ *and* $\tau_2$ *be extensions that are CK-equivalent. If* $\tau_1' \approx_w \tau_1$ *and* $\tau_2' \approx_w \tau_2$, *then* $\tau_1'$ *and* $\tau_2'$ *are CK-equivalent.*

*Proof.* Straightforward. □

*Proof of Lemma 3.3.* Suppose that there exists a partial isometry $v \in \mathcal{Q}$ for which (3.1) holds. If $v^*v = vv^* = 1$, then $\tau_1 \approx_w \tau_2$, and we trivially have $\tau_1 \sim_w \tau_2$. If $v^*v < 1$ and $vv^* < 1$, then there is a partial isometry $u$ with $u^*u = 1 - v^*v$ and $uu^* = 1 - vv^*$. Then $u + v$ is a unitary, and $(v + u)\tau_2(a)(v^* + u^*) = \tau_1(a)$ so $\tau_1 \approx_w \tau_2$ and we again have $\tau_1 \sim_w \tau_2$. So we may as well assume that $v$ is a nonunitary isometry; i.e. $v^*v = 1$ and $vv^* < 1$. Let $\sigma_1 = \pi \circ \hat{\sigma}_1$ be an essential degenerate extension, and let $\sigma_2$ be the essential degenerate extension coming from the compression of $\hat{\sigma}_1$, i.e. $\sigma_2 := \pi \circ \operatorname{Ad}(v) \circ \hat{\sigma}_1$. Let $U \in M_2(\mathcal{Q})$ be given by

$$U = \begin{pmatrix} v^* & 0 \\ 1 - vv^* & v \end{pmatrix}.$$

Then $U$ is a unitary and $U(\tau_1 \oplus \sigma_1)U^* = \tau_2 \oplus \sigma_2$. It follows that $\tau_1 + \sigma_1 \approx_w \tau_2 + \sigma_2$, and $\tau_1 \sim_w \tau_2$. Thus we have shown that (3.1) implies weak stable equivalence.

Now assume that $\tau_1 \sim_w \tau_2$. Suppose that $\tau_1 + \lambda_1 \approx_w \tau_2 + \lambda_2$ with each $\lambda_i$ degenerate, and $\lambda_i = \pi \circ \hat{\lambda}_i$. If both $\tau_1$ and $\tau_2$ are nonunital, then both are absorbing by [3, Theorem 15.12.3]. Consequently, $\tau_1 \approx_w \tau_2$ and (3.1) certainly holds. Now suppose that $\tau_1$ is unital and $\tau_2$ is not. Let $v$ be a nonunitary isometry in $\mathcal{Q}$, and define

$$\tau_1'(a) := v\tau_1(a)v^* \text{ for all } a \in A.$$

Since $\tau_1$ and $\tau_1'$ are CK-equivalent, $\tau_1' + \sigma_1 \approx_w \tau_1 + \sigma_2$, where $\sigma_1$ and $\sigma_2$ are as above. Since $\sim_w$ is transitive, $\tau_1' \sim_w \tau_2$. Furthermore, because neither $\tau_1'$ nor $\tau_2$ is unital we know that they are absorbing by [3, Theorem 15.12.3] and thus there is a unitary $u$ such that $\tau_1' = \mathrm{Ad}(u) \circ \tau_2$. Thus

$$\tau_2(a) = u^*v\tau_1(a)v^*u \quad \text{and} \quad \tau_1(a) = v^*u\tau_2(a)u^*v.$$

Because $u$ is a unitary, it follows that $u^*v$ is a partial isometry and (3.1) holds. It only remains to consider the case that both $\tau_i$'s are unital. We let $u$ be a unitary in $\mathcal{Q}$ such that

$$u(\tau_1 + \lambda_1)u^* = \tau_2 + \lambda_2.$$

Let $\lambda$ be a degenerate extension of $A$ by $\mathcal{K}$ that lifts to a unital homomorphism. Since [3, Theorem 15.12.3] implies that each $\tau_i$ is unital-absorbing, it follows that $\tau_1 + \lambda \approx \tau_1$ and $\tau_2 \approx \tau_2 + \lambda$. Thus it suffices to show that $\tau_1 + \lambda$ is CK-equivalent to $\tau_2 + \lambda$. To do this, notice by Lemma 3.4 there are isometries $w_i \in \mathcal{Q}$ such that

$$\lambda = \mathrm{Ad}(w_i^*) \circ \lambda_i.$$

It follows that there are isometries $v_i \in \mathcal{Q}$ such that

$$\tau_i + \lambda = \mathrm{Ad}(v_i^*) \circ (\tau_i + \lambda_i).$$

Notice that

$$w_iw_i^* = (\tau_i + \lambda_i)(1) \quad \text{and} \quad u((\tau_1 + \lambda_1)(1))u^* = (\tau_2 + \lambda_2)(1).$$

Now we compute that

$$\tau_1 + \lambda = \mathrm{Ad}(v_1^*) \circ (\tau_1 + \lambda_1) = \mathrm{Ad}(v_1^*u^*) \circ (\tau_2 + \lambda_2) = \mathrm{Ad}(v_1^*u^*v_2) \circ (\tau_2 + \lambda).$$

Therefore it will suffice to observe that $v_1^*u^*v_2$ is a coisometry. But

$$v_1^*u^*v_2v_2^*uv_1 = v_1^*u^*((\tau_2 + \lambda)(1))uv_1 = v_1^*((\tau_1 + \lambda_1)(1))v_1 = v_1^*v_1v_1^*v_1 = 1.$$

□

In light of this lemma we may think of the class of $\tau$ in $\mathrm{Ext}_w^e(A)$ as the class generated by the relation in (3.1). Furthermore, we see that any two essential degenerate extensions will be equivalent.

For extensions $\tau_1$ and $\tau_2$ we say that $\tau_1 \perp \tau_2$ if there are orthogonal projections $p_1$ and $p_2$ such that $\tau_i(A) \subseteq p_i\mathcal{Q}p_i$. In this case we may define a map $\tau_1 \boxplus \tau_2$ by $a \mapsto \tau_1(a) + \tau_2(a)$. The orthogonality of the projections is enough to ensure that this map will be multiplicative and therefore $\tau_1 \boxplus \tau_2$ will be a homomorphism. The notation $\boxplus$ is used because a quite different meaning has already been assigned to $\tau_1 + \tau_2$.

Now suppose that $\tau_1, \tau_2 \in \mathrm{Ext}_w^e(A)$. We may choose two isometries $v_1, v_2 \in \mathcal{B}(\mathcal{H})$ with orthogonal ranges. If we define $\tau_i' = \mathrm{Ad}(\pi(v_i)) \circ \tau_i$, then $\tau_1'$ and $\tau_2'$ are homomorphisms with $\tau_1' \perp \tau_2'$. Furthermore, $\tau_1' \oplus \tau_2'$ will be unitarily equivalent to $\tau_1' \boxplus \tau_2' \oplus 0$. Consequently $\tau_1' + \tau_2' \sim \tau_1' \boxplus \tau_2'$. Since stable equivalence classes coincide with weak stable equivalence classes, it follows that $\tau_1'+\tau_2' \sim_w \tau_1' \boxplus \tau_2'$. Furthermore, since $\tau_i'$ is CK-equivalent to $\tau_i$ it follows from Lemma 3.3 that $\tau_i' \sim_w \tau_i$. Thus $\tau_1+\tau_2$ and $\tau_1' \boxplus \tau_2'$ define the same class in $\mathrm{Ext}_w^e(A)$.

This gives us Cuntz and Krieger's description of $\mathrm{Ext}(A)$. Provided that there exists an essential degenerate extension of $A$ by $\mathcal{K}$, we may identify $\mathrm{Ext}(A)$ with $\mathrm{Ext}_w^e(A)$ which we then view as the equivalence classes of essential extensions generated by the relation in (3.1). For any two elements $\tau_1, \tau_2 \in \mathrm{Ext}_w^e(A)$, we define their sum to be $\tau_1 + \tau_2 = \tau_1' \boxplus \tau_2'$ where $\tau_1'$ and $\tau_2'$ are essential extensions such that $\tau_1' \perp \tau_2'$ and $\tau_i' \sim_w \tau_i$. Note that the common class of all degenerate essential extensions acts as the neutral element in $\mathrm{Ext}_w^e(A)$.

## 4. Graph $C^*$-Algebra Preliminaries

A (directed) graph $G = (G^0, G^1, r, s)$ consists of a countable set $G^0$ of vertices, a countable set $G^1$ of edges, and maps $r, s : G^1 \to G^0$ that identify the range and source of each edge. A vertex $v \in G^0$ is called a *sink* if $s^{-1}(v) = \emptyset$ and a *source* if $r^{-1}(v) = \emptyset$. All of our graphs will be assumed to be *row-finite* in that each vertex emits only finitely many edges

If $G$ is a row-finite directed graph, a *Cuntz-Krieger $G$-family* in a $C^*$-algebra is a set of mutually orthogonal projections $\{p_v : v \in G^0\}$ together with a set of partial isometries $\{s_e : e \in G^1\}$ that satisfy the *Cuntz-Krieger relations*

$$s_e^* s_e = p_{r(e)} \text{ for } e \in E^1 \quad \text{and} \quad p_v = \sum_{\{e : s(e) = v\}} s_e s_e^* \text{ whenever } v \in G^0 \text{ is not a sink.}$$

Then $C^*(G)$ is defined to be the $C^*$-algebra generated by a universal Cuntz-Krieger $G$-family [9, Theorem 1.2].

A *path* in a graph $G$ is a finite sequence of edges $\alpha := \alpha_1 \alpha_2 \dots \alpha_n$ for which $r(\alpha_i) = s(\alpha_{i+1})$ for $1 \leq i \leq n-1$, and we say that such a path has length $|\alpha| = n$. For $v, w \in G^0$ we write $v \geq w$ to mean that there exists a path with source $v$ and range $w$. For $K, L \subseteq G^0$ we write $K \geq L$ to mean that for each $v \in K$ there exists $w \in L$ such that $v \geq w$.

A *loop* is a path whose range and source are equal. An *exit* for a loop $x := x_1 \dots x_n$ is an edge $e$ for which $s(e) = s(x_i)$ for some $i$ and $e \neq x_i$. A graph is said to satisfy Condition (L) if every loop in $G$ has an exit.

If $G$ is a graph then we may associate two matrices to $G$. The *vertex matrix* of $G$ is the $G^0 \times G^0$ matrix $A_G$ whose entries are given by $A_G(v, w) := \#\{e \in G^1 : s(e) = v \text{ and } r(e) = w\}$. The *edge matrix* of $G$ is the $G^1 \times G^1$ matrix $B_G$ whose entries are given by

$$B_G(e, f) := \begin{cases} 1 & \text{if } r(e) = s(f). \\ 0 & \text{otherwise.} \end{cases}$$

Notice that if $G$ is a row-finite graph, then the rows of both $A_G$ and $B_G$ will eventually be zero. Hence left multiplication gives maps $A_G : \prod_{G^0} \mathbb{Z} \to \prod_{G^0} \mathbb{Z}$ and $B_G : \prod_{G^1} \mathbb{Z} \to \prod_{G^1} \mathbb{Z}$. Also the maps $A_G - I : \prod_{G^0} \mathbb{Z} \to \prod_{G^0} \mathbb{Z}$ and $B_G - I : \prod_{G^1} \mathbb{Z} \to \prod_{G^1} \mathbb{Z}$ will prove important in later portions of this paper.

## 5. The Ext Group for $C^*(G)$

**Lemma 5.1.** *Suppose that $p_1, p_2, \dots$ is a countable sequence of pairwise orthogonal projections in $\mathcal{Q}$. Then there are pairwise orthogonal projections $P_1, P_2, \dots$ in $\mathcal{B}$ such that $\pi(P_i) = p_i$ for $i = 1, 2, \dots$.*

*Proof.* We first show how to find $P_1 \in \mathcal{B}$ such that $\pi(P_1) = p_1$. Lift $p$ to a self-adjoint element $T \in \mathcal{B}$. Then $\pi(T^2 - T) = 0$ and $T^2 - T$ is compact. Therefore $\sigma(T)$ is a countable set whose only accumulation point is 0. In particular, there exists $a \in (0,1)$ and $\epsilon > 0$ such that $(a - \epsilon, a + \epsilon) \cap \sigma(T) = \emptyset$. Thus we may use functional calculus to create $P_1$.

We now give a recursive definition for the other $P_i$'s. Suppose that $P_1, \ldots, P_n$ are pairwise orthogonal lifts of $p_1, \ldots, p_n$ to projections in $\mathcal{B}$. Let $P'_{n+1}$ be any lift of $p_{n+1}$ to a projection. Let $P''_{n+1} := (1 - P_1 - \ldots - P_n)P'_{n+1}(1 - P_1 - \ldots - P_n)$. Then $\pi(P''_{n+1}) = p_{n+1}$, $P''_{n+1}P_i = 0$ if $i = 1, 2, \ldots, n$, and $P''_{n+1}$ is self-adjoint. As in the previous paragraph, we may use the functional calculus to obtain a continuous function $f$ for which $P_{n+1} := f(P''_{n+1})$ is a projection, $f(0) = 0$, and $\pi(P_{n+1}) = p_{n+1}$. Since $P_{n+1}$ can be approximated by polynomials in $P''_{n+1}$ with no constant terms, it follows that $P_{n+1}P_i = 0$ for all $1 \leq i \leq n$. Taking adjoints implies that $P_i P_{n+1} = 0$ as well. □

The following comes from [6, Lemma V.6.4].

**Lemma 5.2.** *If $w$ is a partial isometry in $\mathcal{Q}$, then there exists a partial isometry $V$ in $\mathcal{B}$ such that $\pi(V) = w$.*

**Lemma 5.3.** *If $w$ is a unitary in $\mathcal{Q}$, then $w$ can be lifted to either an isometry or coisometry $U \in \mathcal{B}$.*

*Proof.* By Lemma 5.2 we may choose a partial isometry $V \in \mathcal{B}$ for which $\pi(V) = w$. Let $P = V^*V$ and $Q = VV^*$. Because $1 - P$ and $1 - Q$ are compact projections, it follows that $1 - P$ and $1 - Q$ have finite rank. Replacing $w$ by $w^*$ if necessary, we may assume that rank $(1 - P) \leq$ rank $(1 - Q)$. Let $V_0$ be any partial isometry with source projection $V_0^*V_0 = 1 - P$ and range projection $V_0V_0^* \leq 1 - Q$. Then $U := V + V_0$ is an isometry and since $V_0 = V_0V_0^*V_0 = V_0(1 - P)$ is compact, we see that $\pi(U) = \pi(V + V_0) = \pi(V) = w$. □

For the rest of this section let $G$ be a row-finite graph with no sinks that satisfies Condition (L). Since $C^*(G)$ is separable, there will exist an essential degenerate extension of $C^*(G)$ [3, §15.5]. (In fact, we shall prove that there are many essential degenerate extensions in Lemma 5.7.) Therefore we may use Cuntz and Krieger's description of Ext discussed in §3.

Let $E \in \mathcal{Q}$ be a projection. By Lemma 5.1 we know that there exists a projection $E' \in \mathcal{B}$ such that $\pi(E') = E$. If $X$ is an element of $\mathcal{Q}$ such that $EXE$ is invertible in $E\mathcal{Q}E$, then we denote by $\text{ind}_E(X)$ the Fredholm index of $E'X'E'$ in im $E'$, where $X' \in \mathcal{B}$ is such that $\pi(X') = X$. Since the Fredholm index is invariant under compact perturbations, this definition does not depend on the choice of $E'$ or $X'$. The following two lemmas are taken from [5].

**Lemma 5.4.** *Let $E, F \in \mathcal{Q}$ be orthogonal projections, and let $X$ be an element of $\mathcal{Q}$ such that $EXE$ and $FXF$ are invertible in $E\mathcal{Q}E$ and $F\mathcal{Q}F$ and such that $X$ commutes with $E$ and $F$. Then* $\text{ind}_{E+F}(X) = \text{ind}_E(X) + \text{ind}_F(X)$.

**Lemma 5.5.** *Let $X$ and $Y$ be invertible operators in $E\mathcal{Q}E$. Then* $\text{ind}_E(XY) = \text{ind}_E(X) + \text{ind}_E(Y)$.

In addition, we shall make use of the following lemmas to define a map from $\text{Ext}(C^*(G))$ into $\text{coker}(B_G - I)$. The first lemma is an immediate consequnce of the Gauge-Invariant Uniqueness Theorem for graph algebras [2, Theorem 3.1].

**Lemma 5.6.** *Let $G$ be a graph that satisfies Condition (L), and let $\{s_e, p_v\}$ be the canonical Cuntz-Krieger $G$-family in $C^*(G)$. If $I$ is an ideal of $C^*(G)$ with the property that $p_v \notin I$ for all $v \in G^0$, then $I = \{0\}$.*

**Lemma 5.7.** *Let $G$ be a row-finite graph with no sinks that satisfies Condition (L), and let $\tau : C^*(G) \to \mathcal{Q}$ be an essential extension of $C^*(G)$. If $\{s_e, p_v\}$ is the canonical Cuntz-Krieger $G$-family, then there exists a degenerate essential extension $t : C^*(G) \to \mathcal{Q}$ such that $t(s_e s_e^*) = \tau(s_e s_e^*)$ for all $e \in G^1$.*

*Proof.* Since $\tau$ is essential, $\{\tau(s_e s_e^*)\}_{e\in G^1}$ is a countable set of mutually orthogonal nonzero projections and we may use Lemma 5.1 to lift them to a collection $\{R_e\}_{e\in G^1}$ of mutually orthogonal nonzero projections in $\mathcal{B}$. Now each $\mathcal{H}_e := \text{im } R_e$ is infinite-dimensional, and for each $v \in G^0$ we define $\mathcal{H}_v = \bigoplus_{\{s(e)=v\}} \mathcal{H}_e$. Then each $\mathcal{H}_v$ is infinite-dimensional and for each $e \in G^1$ we can let $T_e$ be a partial isometry with initial space $\mathcal{H}_{r(e)}$ and final space $\mathcal{H}_e$. Also for each $v \in G^0$ we shall let $Q_v$ be the projection onto $\mathcal{H}_v$. Then $\{T_e, Q_v\}$ is a Cuntz-Krieger $G$-family. By the universal property of $C^*(G)$ there exists a homomorphism $\tilde{t} : C^*(G) \to \mathcal{B}$ such that $\tilde{t}(p_v) = Q_v$ and $\tilde{t}(s_e) = T_e$. Let $t := \pi \circ \tilde{t}$. Then $t$ is a degenerate extension and $t(s_e s_e^*) = \pi(\tilde{t}(s_e s_e^*)) = \pi(T_e T_e^*) = \pi(R_e) = \tau(s_e s_e^*)$. Furthermore, for all $v \in G^0$ we have that

$$t(p_v) = \sum_{s(e)=v} t(s_e s_e^*) = \sum_{s(e)=v} \tau(s_e s_e^*) = \tau(p_v) \neq 0$$

so $t$ is essential. □

*Remark* 5.8. Suppose that $G$ is a graph with no sinks, $\tau$ is an extension of $C^*(G)$, and $t$ is another extension for which $t(s_e s_e^*) = \tau(s_e s_e^*)$. Then $t$ will also have the property that $t(p_v) = t(\sum s_e s_e^*) = \sum t(s_e s_e^*) = \sum \tau(s_e s_e^*) = \tau(\sum s_e s_e^*) = \tau(p_v)$ for any $v \in G^0$.

*Definition* 5.9. Let $\tau : C^*(G) \to \mathcal{Q}$ be an essential extension of $C^*(G)$, and for each $e \in G^1$ define $E_e := \tau(s_e s_e^*)$. If $t : C^*(G) \to \mathcal{Q}$ is another essential extension of $C^*(G)$ with the property that $t(s_e s_e^*) = E_e$, then we define a vector $d_{\tau,t} \in \prod_{G^1} \mathbb{Z}$ by

$$d_{\tau,t}(e) = -\text{ind}_{E_e} \tau(s_e) t(s_e^*).$$

Note that this is well-defined since $E_e \tau(s_e) t(s_e^*) E_e = \tau(s_e) t(s_e^*)$ and by Remark 5.8 we have that $\tau(s_e) t(s_e^*) \tau(s_e^*) t(s_e) = \tau(s_e) \tau(s_e^* s_e) \tau(s_e^*) = E_e$ so $\tau(s_e) t(s_e^*)$ is invertible in $E_e \mathcal{Q} E_e$.

*Remark* 5.10. If $E \in \mathcal{Q}$ is a projection and $E' \in \mathcal{B}$ is a lift of $E$ to a projection in $\mathcal{B}$, then one can see that $\mathcal{Q}(E'(\mathcal{H})) \cong E\mathcal{Q}E$ via the obvious correspondence. In the rest of this paper we shall often identify $\mathcal{Q}(E'(\mathcal{H}))$ with $E\mathcal{Q}E$.

**Lemma 5.11.** *Let $E \in \mathcal{Q}$ be a projection and $X \in \mathcal{Q}$, and suppose that $EXE$ is invertible in $E\mathcal{Q}E$. If $V \in \mathcal{Q}$ is a partial isometry with initial projection $V^*V = E$ and final projection $VV^* = F$, then* $\text{ind}_E X = \text{ind}_F VXV^*$.

*Proof.* By Lemma 5.2 there exists a partial isometry $W \in \mathcal{B}$ such that $\pi(W) = V$. Let $E' := W^*W$ and $F' := WW^*$ Also choose any $X' \in \mathcal{B}$ such that $\pi(X') = X$. Define $\Phi_V : \mathcal{B}(E'(\mathcal{H})) \to \mathcal{B}(F'(\mathcal{H}))$ by

$$\Phi_V(T) = WTW^*.$$

We shall show that $\Phi_V$ is a homomorphism. To see that $\Phi_V$ is injective let $\Phi_V(T) = 0$. Then

$$T = E'TE' = W^*WTW^*W = W^*\Phi_V(T)W = 0$$

and thus $\ker \Phi_V = \{0\}$.

In addition, if $T \in \mathcal{B}(F'(\mathcal{H}))$, then

$$E'(W^*FW)E' = W^*WW^*FWW^*W = W^*FW$$

and $W^*FW \in \mathcal{B}(E'(\mathcal{H}))$. Because $\Phi_V(W^*FW) = F$ this implies that $\Phi_V$ is surjective. Therefore, $\Phi_V : \mathcal{B}(E'(\mathcal{H})) \to \mathcal{B}(F'(\mathcal{H}))$ is an isomorphism.

Because $\Phi_V$ is an isomorphism, the fact that $EXE$ is an invertible operator in $\mathcal{Q}(E'(\mathcal{H})E') \cong E\mathcal{Q}E$ implies that the Fredholm index of $E'XE'$ in $\mathcal{B}(E'(\mathcal{H}))$ is the same as the Fredholm index of $\Phi_V(E'XE') = FWXW^*F'$ in $\mathcal{B}(F'(\mathcal{H}))$. Therefore $\operatorname{ind}_E X = \operatorname{ind}_F VXV^*$. ☐

**Proposition 5.12.** *Let $G$ be a row-finite graph with no sinks that satisfies Condition (L). Also let $\tau$ be an essential extension of $C^*(G)$ and $E_e := \tau(s_e s_e^*)$ for $e \in G^1$. If $t$ and $t'$ are essential extensions of $C^*(G)$ that are CK-equivalent and satisfy $t(s_e s_e^*) = t'(s_e s_e^*) = E_e$, then $d_{\tau,t} - d_{\tau,t'} \in \operatorname{im}(B_G - I)$.*

*Proof.* Since $t$ and $t'$ are CK-equivalent, there exists a partial isometry $U \in \mathcal{Q}$ such that $t = \operatorname{Ad}(U)\circ t'$ and $t' = \operatorname{Ad}(U^*)\circ t$. Now notice that $U$ commutes with $E_e$. Thus for any $e \in G^1$ we have $\tau(s_e s_e^*) = \sum_{s(f)=r(e)} \tau(s_f s_f^*) = \sum_{s(f)=r(e)} t(s_f s_f^*) = t(s_e^* s_e)$ and

$$\begin{aligned}
d_{\tau,t}(e) - d_{\tau,t'}(e) &= -\operatorname{ind}_{E_e} \tau(s_e)t(s_e^*) + \operatorname{ind}_{E_e} \tau(s_e)t'(s_e^*) \\
&= \operatorname{ind}_{E_e} t(s_e)\tau(s_e^*) + \operatorname{ind}_{E_e} \tau(s_e)t'(s_e^*) \\
&= \operatorname{ind}_{E_e} t(s_e)\tau(s_e^* s_e)t'(s_e^*) \qquad \text{by Lemma 5.5} \\
&= \operatorname{ind}_{E_e} t(s_e)t'(s_e^*) \\
&= -d_{t,t'}(e).
\end{aligned}$$

Hence $d_{\tau,t} - d_{\tau,t'} = -d_{t,t'}$. Now let $k \in \prod_{G^1} \mathbb{Z}$ be the vector given by $k(f) := \operatorname{ind}_{E_f} U$. Then for any $e \in G^1$ we have

$$\begin{aligned}
d_{t,t'}(e) &= -\operatorname{ind}_{E_e} t(s_e)t'(s_e^*) \\
&= -\operatorname{ind}_{E_e} t(s_e)Ut(s_e^*)U^* \\
&= -\operatorname{ind}_{E_e} t(s_e)Ut(s_e^*) - \operatorname{ind}_{E_e} U^* \qquad \text{by Lemma 5.5} \\
&= -\operatorname{ind}_{t(s_e^* s_e)} U - \operatorname{ind}_{E_e} U^* \qquad \text{by Lemma 5.11} \\
&= -\operatorname{ind}_{\sum\limits_{s(f)=r(e)} E_f} U + \operatorname{ind}_{E_e} U \\
&= -\sum_{s(f)=r(e)} \operatorname{ind}_{E_f} U + \operatorname{ind}_{E_e} U \qquad \text{by Lemma 5.4} \\
&= -\left( \sum_{f \in G^1} B_G(e,f)k(f) - k(e) \right)
\end{aligned}$$

so $d_{t,t'} = -(B_G - I)k$ and $d_{\tau,t} - d_{\tau,t'} = -d_{t,t'} \in \operatorname{im}(B_G - I)$. ☐

*Definition* 5.13. Let $G$ be a row-finite graph with no sinks that satisfies Condition (L). Let $B_G$ be the edge matrix of $G$ and $B_G - I : \prod_{G^1} \mathbb{Z} \to \prod_{G^1} \mathbb{Z}$. If $\tau$ is an essential extension of $C^*(G)$, then we shall define an element $d_\tau \in \operatorname{coker}(B_G - I)$ by

$$d_\tau := [d_{\tau,t}] \in \operatorname{coker}(B_G - I),$$

where $t$ is any degenerate extension with the property that $t(s_e s_e^*) = \tau(s_e s_e^*)$ for all $e \in G^1$.

In the above definition, the existence of $t$ follows from Lemma 5.7. In addition, since any two degenerate essential extensions are CK-equivalent, it follows from Proposition 5.12 that the class of $d_{\tau,t}$ in $\operatorname{coker}(B_G - I)$ will be independent of the choice of $t$. Therefore $d_\tau$ is well-defined.

**Lemma 5.14.** *Suppose that $\tau_1$ and $\tau_2$ are extensions of a $C^*$-algebra $A$, and that $v$ is a partial isometry in $\mathcal{Q}$ for which $\tau_1 = \operatorname{Ad}(v) \circ \tau_2$ and $\tau_2 = \operatorname{Ad}(v^*) \circ \tau_1$. Then there exists either an isometry or coisometry $W \in \mathcal{B}$ such that $\tau_1 = \operatorname{Ad}\pi(W) \circ \tau_2$ and $\tau_2 = \operatorname{Ad}\pi(W^*) \circ \tau_1$.*

*Proof.* Since $v$ is a partial isometry Lemma 5.2 tells us that there exists a partial isometry $V \in \mathcal{B}$ such that $\pi(V) = v$. If we consider the projections $1 - V^*V$ and $1 - VV^*$, then one of these projections has a rank greater than or equal to the rank of the other.

Let us suppose first that the rank of $1 - VV^*$ is greater than or equal to the rank of $1 - V^*V$. Then we may choose a partial isometry $V_0$ in $\mathcal{B}$ with source projection $V_0^*V_0 = 1 - V^*V$ and range projection $V_0V_0^* \leq 1 - VV^*$. If we define $W = V + V_0$, then $W$ is an isometry. If we now let $w := \pi(W)$, then $wv^*v = \pi(WV^*V) = \pi((V + V_0)V^*V) = \pi(VV^*V) = \pi(V) = v$. Thus $\operatorname{Ad}(w) \circ \tau_2 = \operatorname{Ad}(w) \circ \operatorname{Ad}(v^*) \circ \tau_1 = \operatorname{Ad}(w) \circ \operatorname{Ad}(v^*) \circ \operatorname{Ad}(v) \circ \tau_2 = \operatorname{Ad}(wv^*v) \circ \tau_2 = \operatorname{Ad}(v) \circ \tau_2 = \tau_1$. A similar argument shows that $\operatorname{Ad}(w^*) \circ \tau_1 = \tau_2$.

On the other hand, if it is the case that the rank of $1 - VV^*$ is less than the rank of $1 - V^*V$, then we may choose a partial isometery $V_0$ in $\mathcal{B}$ with source projection $V_0^*V_0 = 1 - VV^*$ and range projection $V_0V_0^* \leq 1 - V^*V$. Then $W = V + V_0^*$ will be a coisometry, and a calculation similar to the one above shows that $v$ may be replaced by $w = \pi(W)$. ☐

**Corollary 5.15.** *Let $\tau_1$ and $\tau_2$ be essential extensions of a $C^*$-algebra $A$. Then $\tau_1$ and $\tau_2$ are CK-equivalent if and only if there exists either an isometry or coisometry $W$ in $\mathcal{B}$ such that $\tau_1 = \operatorname{Ad}\pi(W) \circ \tau_2$ and $\tau_2 = \operatorname{Ad}\pi(W^*) \circ \tau_1$.*

**Lemma 5.16.** *Let $G$ be a row-finite graph with no sinks that satisfies Condition (L). Suppose that $\tau_1$ and $\tau_2$ are two essential extensions of $C^*(G)$ that are equal in $\operatorname{Ext}(C^*(G))$. Then $d_{\tau_1}$ and $d_{\tau_2}$ are equal in $\operatorname{coker}(B_G - I)$.*

*Proof.* Since $\tau_1$ and $\tau_2$ are equal in $\operatorname{Ext}(C^*(G))$ it follows that they are CK-equivalent. By interchanging $\tau_1$ and $\tau_2$ if necessary, we may use Corollary 5.15 to choose an isometry $W$ in $\mathcal{B}$ for which $\tau_1 = \operatorname{Ad}\pi(W) \circ \tau_2$ and $\tau_2 = \operatorname{Ad}\pi(W^*) \circ \tau_1$. For each $e \in G^1$ define $E_e := \tau_1(s_e s_e^*)$ and $F_e := \tau_2(s_e s_e^*)$. By Lemma 5.7 there exists a degenerate essential extension $t_2 = \pi \circ \tilde{t}_2$ with the property that $t_2(s_e s_e^*) = \tau_2(s_e s_e^*) = F_e$ for all $e \in G^1$. Then $\tilde{t}_1 := W\tilde{t}_2W^*$ will be a representation of $C^*(G)$ ($\tilde{t}_1$ is multiplicative since $W$ is an isometry), and thus $t_1 := \pi \circ \tilde{t}_1$ will be a degenerate extension with the property that $t_1(s_e s_e^*) = \tau_1(s_e s_e^*)$. Now since $\tau_1$ is essential we have that

$$t_1(p_v) = \sum_{s(e)=v} t_1(s_e s_e^*) = \sum_{s(e)=v} \tau_1(s_e s_e^*) = \tau_1(p_v) \neq 0.$$

Therefore $p_v \notin \ker t_1$ for all $v \in G^0$ and it follows from Lemma 5.6 that $\ker t_1 = \{0\}$, and thus $t_1$ is essential.

Now recall that $E_e := \tau_1(s_e s_e^*)$ and $F_e := \tau_2(s_e s_e^*)$. Since $W$ is an isometry, we see that $\pi(W)F_e$ is a partial isometry with source projection $F_e$ and range projection $E_e$. Therefore by Lemma 5.11 it follows that

$$\begin{aligned}
\operatorname{ind}_{F_e} \tau_2(s_e)t_2(s_e^*) &= \operatorname{ind}_{E_e} \pi(W)F_e\tau_2(s_e)t_2(s_e^*)F_e\pi(W^*) \\
&= \operatorname{ind}_{E_e} \pi(W)\tau_2(s_e)t_2(s_e^*)\pi(W^*) \\
&= \operatorname{ind}_{E_e} \pi(W)\tau_2(s_e)\pi(W^*)\pi(W)t_2(s_e^*)\pi(W^*) \\
&= \operatorname{ind}_{E_e} \tau_1(s_e)t_1(s_e^*)
\end{aligned}$$

and $d_{\tau_2}$ equals $d_{\tau_1}$ in $\operatorname{coker}(B_G - I)$. □

*Definition* 5.17. If $G$ is a row-finite graph with no sinks that satisfies Condition (L), we define the *Cuntz-Krieger map* to be the map $d : \operatorname{Ext}(C^*(G)) \to \operatorname{coker}(B_G - I)$ defined by $\tau \mapsto d_\tau$.

The previous lemma shows that the Cuntz-Krieger map $d$ is well-defined, and the next lemma shows that it is a homomorphism.

**Lemma 5.18.** *Suppose that $G$ is a row-finite graph with no sinks that satisfies Condition (L). Then the Cuntz-Krieger map is additive.*

*Proof.* Let $\tau_1$ and $\tau_2$ be elements of $\operatorname{Ext}(C^*(G))$ and choose the representatives $\tau_1$ and $\tau_2$ such that $\tau_1 \perp \tau_2$. Let $t_1$ and $t_2$ be degenerate essential extensions such that $t_1(s_e s_e^*) = \tau_1(s_e s_e^*)$ and $t_2(s_e s_e^*) = \tau_2(s_e s_e^*)$.

Because $\tau_1 \perp \tau_2$ we know that there exist orthogonal projections $p_1, p_2 \in \mathcal{Q}$ such that $\tau_i(C^*(G)) \subseteq p_i\mathcal{Q}p_i$ for $i \in \{1, 2\}$. Now for any $e \in E^1$ and $i \in \{1, 2\}$ we have that

$$\begin{aligned}
p_i t_i(s_e) p_i &= p_i t_i(s_e s_e^*) t_i(s_e) t_i(s_e^* s_e) p_i = p_i \tau_i(s_e s_e^*) t_i(s_e) \tau_i(s_e^* s_e) p_i \\
&= \tau_i(s_e s_e^*) t_i(s_e) \tau_i(s_e^* s_e) = t_i(s_e s_e^*) t_i(s_e) t_i(s_e^* s_e) = t_i(s_e)
\end{aligned}$$

Thus $t_i(s_e) \in p_i\mathcal{Q}p_i$ for all $e \in E^1$. Since the $s_e$'s generate $C^*(G)$ it follows that $t_i(C^*(G)) \subseteq p_i\mathcal{Q}p_i$. Thus $t_1 \perp t_2$, and we may form the essential extension $t_1 \boxplus t_2$ given by $a \mapsto t_1(a) + t_2(a)$.

Notice that $t_1$ and $t_2$ are degenerate extensions, and thus $t_1 + t_2$ is a degenerate extension. Because $t := t_1 \boxplus t_2$ is weakly stably equivalent to $t_1 + t_2$ we see that $t$ is in the zero class in Ext. But since $t$ is an essential extension with the property that $t(s_e s_e^*) = \tau_1(s_e s_e^*) + \tau_2(s_e s_e^*)$, it follows from Lemma 5.12 that $d_{\tau \boxplus \tau_2} = [d_{\tau_1 \boxplus \tau_2, t}]$ in $\operatorname{coker}(B_G - I)$. Furthermore, since $\operatorname{ind}_E X = \operatorname{ind}_E EX = \operatorname{ind}_E XE$, we have that

$$\begin{aligned}
d_{\tau_1 \boxplus \tau_2, t}(e) &= -\operatorname{ind}_{(\tau_1 \boxplus \tau_2)(s_e s_e^*)} (\tau_1 \boxplus \tau_2)(s_e) t(s_e^*) \\
&= -\operatorname{ind}_{\tau_1(s_e s_e^*)} (\tau_1 \boxplus \tau_2)(s_e) t(s_e^*) \\
&\qquad\qquad - \operatorname{ind}_{\tau_2(s_e s_e^*)} (\tau_1 \boxplus \tau_2)(s_e) t(s_e^*) \qquad \text{(by Lemma 5.4)} \\
&= -\operatorname{ind}_{\tau_1(s_e s_e^*)} \tau_1(s_e s_e^*)(\tau_1(s_e) + \tau_2(s_e)) t(s_e^*) \\
&\qquad\qquad - \operatorname{ind}_{\tau_2(s_e s_e^*)} \tau_2(s_e s_e^*)(\tau_1(s_e) + \tau_2(s_e)) t(s_e^*) \\
&= -\operatorname{ind}_{\tau_1(s_e s_e^*)} \tau_1(s_e) t(s_e^*) - \operatorname{ind}_{\tau_2(s_e s_e^*)} \tau_2(s_e) t(s_e^*) \\
&= -\operatorname{ind}_{\tau_1(s_e s_e^*)} \tau_1(s_e)(t_1(s_e^*) + t_2(s_e^*)) \tau_1(s_e s_e^*) \\
&\qquad\qquad - \operatorname{ind}_{\tau_2(s_e s_e^*)} \tau_2(s_e)(t_1(s_e^*) + t_2(s_e^*)) \tau_2(s_e s_e^*)
\end{aligned}$$

$$\begin{aligned}
&= -\operatorname{ind}_{\tau_1(s_e s_e^*)}\tau_1(s_e)(t_1(s_e^*)+t_2(s_e^*))t_1(s_e s_e^*) \\
&\qquad\qquad -\operatorname{ind}_{\tau_2(s_e s_e^*)}\tau_2(s_e)(t_1(s_e^*)+t_2(s_e^*))t_2(s_e s_e^*) \\
&= -\operatorname{ind}_{\tau_1(s_e s_e^*)}\tau_1(s_e)t_1(s_e^*) - \operatorname{ind}_{\tau_2(s_e s_e^*)}\tau_2(s_e)t_2(s_e^*) \\
&= d_{\tau_1,t_1}(e) + d_{\tau_2,t_2}(e).
\end{aligned}$$

So $d_{\tau_1 \boxplus \tau_2, t} = d_{\tau_1,t} + d_{\tau_2,t}$. Also since $\tau_1 \boxplus \tau_2$ is weakly stably equivalent to $\tau_1 + \tau_2$, Lemma 5.16 implies that we have $d_{\tau_1 \boxplus \tau_2} = d_{\tau_1+\tau_2}$ in $\operatorname{coker}(B_G - I)$. Putting this all together gives $d_{\tau_1+\tau_2} = d_{\tau_1 \boxplus \tau_2} = [d_{\tau_1 \boxplus \tau_2, t}] = [d_{\tau_1,t_1} + d_{\tau_2,t_2}] = [d_{\tau_1,t_1}] + [d_{\tau_2,t_2}] = d_{\tau_1} + d_{\tau_2}$ in $\operatorname{coker}(B_G - I)$. Thus $d$ is additive. ☐

We mention the following two lemmas, both of whose proofs are straightforward.

**Lemma 5.19.** *Let $E \in \mathcal{Q}$ be a projection, and suppose that $T$ is a unitary in $E\mathcal{Q}E$ with $\operatorname{ind}_E T = 0$. If $E' \in \mathcal{B}$ is a projection such that $\pi(E') = E$, then there is a unitary $U \in \mathcal{B}(E'\mathcal{H})$ such that $\pi(U) = T$.*

**Lemma 5.20.** *Suppose that $\mathcal{H}_1, \mathcal{H}_2, \dots$ is a countable collection of pairwise orthogonal subspaces of the Hilbert space $\mathcal{H}$, and for each $i \in \{1, 2, \dots\}$ $V_i$ is an operator in $\mathcal{B}(\mathcal{H}_i)$ with norm 1. If we extend each $V_i$ to all of $\mathcal{H}$ by defining it to be zero on $\mathcal{H}_i^\perp$, then the sum $\sum_{i=1}^\infty V_i$ converges in the strong operator topology on $\mathcal{B}(\mathcal{H})$ to an operator of norm 1.*

**Proposition 5.21.** *Let $G$ be a row-finite graph with no sinks that satisfies Condition (L). Then the Cuntz-Krieger map $d : \operatorname{Ext}(C^*(G)) \to \operatorname{coker}(B_G - I)$ defined by $\tau \mapsto d_\tau$ is injective.*

*Proof.* Let $\tau$ be an essential extension of $C^*(G)$ and suppose that $d_\tau$ equals 0 in $\operatorname{coker}(B_G - I)$. Use Lemma 5.7 to choose a degenerate essential extension $t := \pi \circ \tilde{t}$ of $C^*(G)$ such that $t(s_e s_e^*) = E_e := \tau(s_e s_e^*)$ for all $e \in G^1$. Also let $E'_e := \tilde{t}(s_e s_e^*)$.

By hypothesis, there exists $k \in \prod_{G^1} \mathbb{Z}$ such that $d_{\tau,t} = (B_G - I)k$. Since $\tau$ is essential, for all $e \in G^1$ we must have that $\pi(E'_e) = E_e = \tau(s_e s_e^*) \neq 0$. Since $E'_e$ is a projection, this implies that $\dim(\operatorname{im}(E'_e)) = \infty$. Therefore for each $e \in G^1$ we may choose isometries or coisometries $V_e$ in $\mathcal{B}(E'_e(\mathcal{H}))$ such that $\operatorname{ind}_{E_e} V_e = -k(e)$. Extend each $V_e$ to all of $\mathcal{H}$ by defining it to be zero on $(E'_e(\mathcal{H}))^\perp$. Let $U := \sum_{e \in G^1} V_e$. It follows from Lemma 5.20 that this sum converges in the strong operator topology. Notice that for all $e, f \in G^1$ we have

$$V_f \tilde{t}(s_e s_e^*) = V_f E'_f E'_e = \begin{cases} V_f & \text{if } e = f \\ 0 & \text{otherwise.} \end{cases}$$

Since $U$ commutes with $E'_e$ for all $e \in G^1$, we see that $\pi(U)\tau(s_e)\pi(U^*)t(s_e^*)$ is a unitary in $E_e\mathcal{Q}E_e$. Hence we may consider $\operatorname{ind}_{E_e} \pi(U)\tau(s_e)\pi(U^*)t(s_e^*)$. Using the above identity we see that for each $e \in G^1$ we have

$$\begin{aligned}
\operatorname{ind}_{E_e}\pi(U)\tau(s_e)\pi(U^*)t(s_e^*) &= \operatorname{ind}_{E_e}\pi(U)\tau(s_e s_e^*)\tau(s_e)\pi(U^*)t(s_e^*) \\
&= \operatorname{ind}_{E_e}\pi(U\tilde{t}(s_e s_e^*))\tau(s_e)\pi(U^*)t(s_e^*) \\
&= \operatorname{ind}_{E_e}\pi\left(\sum_{f \in G^1} V_f \tilde{t}(s_e s_e^*)\right)\tau(s_e)\pi(U^*)t(s_e^*) \\
&= \operatorname{ind}_{E_e}\pi(V_e\tilde{t}(s_e s_e^*))\tau(s_e)\pi(U^*)t(s_e^*) \\
&= \operatorname{ind}_{E_e}\pi(V_e)\tau(s_e)t(s_e^*)\Big(t(s_e)\pi(U^*)t(s_e^*)\Big). \qquad (5.1)
\end{aligned}$$

Now since $t(s_e)$ is a partial isometry with source projection

$$t(s_e^* s_e) = \sum_{s(f)=r(e)} t(s_f s_f^*) = \sum_{s(f)=r(e)} E_f$$

and range projection $t(s_e s_e^*) = E_e$, we may use Lemma 5.11 to conclude that

$$\operatorname{ind}_{\sum\limits_{s(f)=r(e)} E_f} \pi(U^*) = \operatorname{ind}_{E_e} t(s_e)\pi(U^*)t(s_e^*).$$

This combined with Lemma 5.4 implies that

$$\begin{aligned}
\operatorname{ind}_{E_e} t(s_e)\pi(U^*)t(s_e^*) &= \sum_{s(f)=r(e)} \operatorname{ind}_{E_f} \pi(U^*) \\
&= \sum_{s(f)=r(e)} \operatorname{ind}_{E_f} E_f \pi\left(\sum_{g\in G^1} V_g^*\right) E_f \\
&= \sum_{s(f)=r(e)} \operatorname{ind}_{E_f} \pi\left(\sum_{g\in G^1} E'_f V_g^* E'_f\right) \\
&= \sum_{s(f)=r(e)} \operatorname{ind}_{E_f} \pi(V_f^*) \\
&= \sum_{s(f)=r(e)} k(f) \\
&= \sum_{f\in G^1} B_G(e,f)k(f). \qquad (5.2)
\end{aligned}$$

Combining (5.1) and (5.2) with Lemma 5.5 gives

$$\operatorname{ind}_{E_e} \pi(U)\tau(s_e)\pi(U^*)t(s_e^*) = \left(\sum_{f\in G^1} B_G(e,f)k(f) - k(e)\right) - d_\tau(e) = 0.$$

Thus by Lemma 5.19 there exists an operator $X_e \in \mathcal{B}$ such that the restriction of $X_e$ to $E'_e(\mathcal{H})$ is a unitary operator and $\pi(X_e) = \pi(U)\tau(s_e)\pi(U^*)t(s_e^*)$. Let $T_e := X_e\tilde{t}(s_e)$. Then $T_e$ is a partial isometry that satisfies $T_eT_e^* = E'_e$ and $T_e^*T_e = \tilde{t}(s_e^*)X_e^*X_e\tilde{t}(s_e) = \tilde{t}(s_e^*s_e) = \tilde{t}(p_{r(e)})$. One can then check that $\{\tilde{t}(p_v), T_e\}$ is a Cuntz-Krieger $G$-family in $\mathcal{B}$. Thus by the universal property of $C^*(G)$ there exists a homomorphism $\tilde{\rho} : C^*(G) \to \mathcal{B}$ such that $\tilde{\rho}(p_v) = \tilde{t}(p_v)$ and $\tilde{\rho}(s_e) = T_e$. Let $\rho := \pi \circ \tilde{\rho}$. Then $\rho$ is a degenerate extension of $C^*(G)$. Furthermore, since $\rho(p_v) = t(p_v) \neq 0$ we see that $p_v \notin \ker \rho$ for all $v \in G^0$. Since $G$ satisfies Condition (L), it follows from Lemma 5.6 that $\ker \rho = \{0\}$ and $\rho$ is a degenerate essential extension. In addition, we see that for each $e \in G^1$

$$\begin{aligned}
\rho(s_e) &= \pi(T_e) \\
&= \pi(X_e\tilde{t}(s_e)) \\
&= \pi(U)\tau(s_e)\pi(U^*)t(s_e^*)t(s_e) \\
&= \pi(U)\tau(s_e)\pi\Big(U^* \sum_{s(g)=r(e)} \tilde{t}(s_gs_g^*)\Big) \\
&= \pi(U)\tau(s_e)\pi\Big(\sum_{f\in G^1} V_f^* \sum_{s(g)=r(e)} E'_g\Big)
\end{aligned}$$

$$\begin{aligned}
&= \pi(U)\tau(s_e)\pi\Big(\sum_{s(g)=r(e)} E'_g \sum_{f\in G^1} V_f^*\Big) \\
&= \pi(U)\tau(s_e)\tau(s_e^*s_e)\pi\Big(\sum_{f\in G^1} V_f^*\Big) \\
&= \pi(U)\tau(s_e)\pi(U^*).
\end{aligned}$$

Thus $\rho(s_e) = \pi(U)\tau(s_e)\pi(U^*)$ for all $e \in G^1$, and since the $s_e$'s generate $C^*(G)$, it follows that $\rho(a) = \pi(U)\tau(a)\pi(U^*)$ for all $a \in C^*(G)$ and hence $\rho = \mathrm{Ad}(\pi(U)) \circ \tau$.

In addition, since the $V_e$'s are either isometries or coisometries on $E'_e(\mathcal{H})$ with finite Fredholm index, it follows that $\pi(V_e^*V_e) = \pi(V_eV_e^*) = \pi(E'_e)$. Therefore, for any $e \in G^1$ we have that

$$\begin{aligned}
\pi(U^*U)\tau(s_e) &= \pi\left(U^* \sum_{f\in G^1} V_f \tilde{t}(s_es_e^*)\right)\tau(s_e) \\
&= \pi\left(U^*V_eE'_e\right)\tau(s_e) \\
&= \pi\left(\sum_{f\in G^1} V_f^*E'_eV_e\right)\tau(s_e) \\
&= \pi(V_e^*V_e)\tau(s_e) \\
&= \pi(E'_e)\tau(s_e) \\
&= \tau(s_es_e^*)\tau(s_e) \\
&= \tau(s_e).
\end{aligned}$$

Again, since the $s_e$'s generate $C^*(G)$, it follows that $\pi(U^*U)\tau(a) = \tau(a)$ for all $a \in C^*(G)$. Similarly, $\tau(a)\pi(U^*U) = \tau(a)$ for all $a \in C^*(G)$. Thus $\pi(U^*)\rho(a)\pi(U) = \pi(U^*U)\tau(a)\pi(U^*U) = \tau(a)$ for all $a \in C^*(G)$ and $\tau = \mathrm{Ad}(\pi(U)^*) \circ \rho$.

Now because the $V_e$'s are all isometries or coisometries on orthogonal spaces, it follows that $U$, and hence $\pi(U)$, is a partial isometry. Therefore, $\tau = \rho$ in $\mathrm{Ext}(C^*(G))$ and since $\rho$ is a degenerate essential extension it follows that $\tau = 0$ in $\mathrm{Ext}(C^*(G))$. This implies that $d$ is injective. □

## 6. The Wojciech Map

In the previous section we showed that if $G$ is a row-finite graph with no sinks that satisfies Condition (L), then the Cuntz-Krieger map $d : \mathrm{Ext}(C^*(G)) \to \mathrm{coker}(B_G - I)$ is a monomorphism. It turns out that $d$ is also surjective; that is, it is an isomorphism. In this section we shall prove this fact, but we shall do it in an indirect way. We show that $\mathrm{coker}(B_G - I)$ is isomorphic to $\mathrm{coker}(A_G - I)$ and then compose $d$ with this isomorphism to get a map from $\mathrm{Ext}(C^*(G))$ into $\mathrm{coker}(A_G - I)$. We call this composition the Wojciech map and we shall show that it, and consequently also $d$, is surjective. For the rest of this paper we will be mostly concerned with the Wojciech map and how it relates to 1-sink extensions defined in [13].

*Definition* 6.1. Let $G$ be a graph. The *source matrix* of $G$ is the $G^0 \times G^1$ matrix given by

$$S_G(v,e) = \begin{cases} 1 & \text{if } s(e) = v \\ 0 & \text{otherwise} \end{cases}$$

and the *range matrix* of $G$ is the $G^1 \times G^0$ matrix given by

$$R_G(e,v) = \begin{cases} 1 & \text{if } r(e) = v \\ 0 & \text{otherwise.} \end{cases}$$

Notice that if $G$ is a row-finite graph, then $S_G$ will have rows that are eventually zero and left multiplication by $S_G$ defines a map $S_G : \prod_{G^1} \mathbb{Z} \to \prod_{G^0} \mathbb{Z}$. Also $R_G$ will always have rows that are eventually zero. (In fact, regardless of any conditions on $G$, $R_G$ will have only one nonzero entry in each row.) Therefore left multiplication by $R_G$ defines a map $R_G : \prod_{G^0} \mathbb{Z} \to \prod_{G^1} \mathbb{Z}$. Furthermore, one can see that

$$R_G S_G = B_G \quad \text{and} \quad S_G R_G = A_G.$$

The following lemma is well known for finite graphs and a proof for $S_G$ restricted to the direct sum $S_G : \bigoplus_{G^1} \mathbb{Z} \to \bigoplus_{G^0} \mathbb{Z}$ is given in [11, Lemma 4.2]. Essentially the same proof goes through if we replace the direct sums by direct products.

**Lemma 6.2.** *Let $G$ be a row-finite graph. The map $S_G : \prod_{G^1} \mathbb{Z} \to \prod_{G^0} \mathbb{Z}$ induces an isomorphism $\overline{S_G} : \operatorname{coker}(B_G - I) \to \operatorname{coker}(A_G - I)$.*

*Proof.* Suppose that $z \in \operatorname{im}(B_G - I)$. Then $z = (B_G - I)u$ for some $u \in \prod_{G^1} \mathbb{Z}$. Then

$$S_G z = S_G(B_G - I)u = S_G(R_G S_G - I)u = (S_G R_G - I)S_G u = (A_G - I)S_G u$$

and $S_G$ does in fact map $\operatorname{im}(B_G - I)$ into $\operatorname{im}(A_G - I)$. Thus $S_G$ induces a homomorphism $\overline{S_G}$ of $\operatorname{coker}(B_G - I)$ into $\operatorname{coker}(A_G - I)$. In the same way, $R_G$ induces a homomorphism $\overline{R_G}$ from $\operatorname{coker}(A_G - I)$ into $\operatorname{coker}(B_G - I)$, which we claim is an inverse for $\overline{S_G}$. We see that

$$\begin{aligned} \overline{R_G} \circ \overline{S_G}(u + \operatorname{im}(B_G - I)) &= R_G S_G u + \operatorname{im}(B_G - I) \\ &= u + (B_G u - u) + \operatorname{im}(B_G - I) \\ &= u + \operatorname{im}(B_G - I) \end{aligned}$$

and similarly $\overline{S_G} \circ \overline{R_G}$ is the identity on $\operatorname{coker}(A_G - I)$. ☐

*Definition* 6.3. Let $G$ be a row-finite graph with no sinks that satisfies Condition (L), and let $d : \operatorname{Ext}(C^*(G)) \to \operatorname{coker}(B_G - I)$ be the Cuntz-Krieger map. The *Wojciech map* is the homomorphism $\omega : \operatorname{Ext}(C^*(G)) \to \operatorname{coker}(A_G - I)$ given by $\omega := \overline{S_G} \circ d$. Given an extension $\tau$ of $C^*(G)$, we shall refer to the class $\omega(\tau)$ in $\operatorname{coker}(A_G - I)$ as the *Wojciech class* of $\tau$.

**Lemma 6.4.** *Let $G$ be a row-finite graph with no sinks that satisfies Condition (L). Then the Wojciech map associated to $G$ is a monomorphism.*

*Proof.* Since $\omega = \overline{S_G} \circ d$, and $\overline{S_G}$ is an isomorphism by Lemma 6.2, the result follows from Proposition 5.21. ☐

We shall eventually show that the Wojciech map is also surjective; that is, it is an isomorphism. In order to do this we consider 1-sink extensions, which were introduced in [13], and describe a way to associate elements of $\operatorname{Ext}(C^*(G))$ to them.

*Definition* 6.5. [13, Definition 1.1] Let $G$ be a row-finite graph. A *1-sink extension* of $G$ is a row-finite graph $E$ that contains $G$ as a subgraph and satisfies:

1. $H := E^0 \setminus G^0$ is finite, contains no sources, and contains exactly 1 sink $v_0$.

2. There are no loops in $E$ whose vertices lie in $H$.
3. If $e \in E^1 \setminus G^1$, then $r(e) \in H$.
4. If $w$ is a sink in $G$, then $w$ is a sink in $E$.

We will write $(E, v_0)$ for the 1-sink extension, where $v_0$ denotes the sink outside $G$.

If $(E, v_0)$ is a 1-sink extension of $G$, then we may let $\pi_E : C^*(E) \to C^*(G)$ be the surjection described in [13, Corollary 1.3]. Then $\ker \pi_E = I_{v_0}$ where $I_{v_0}$ is the ideal in $C^*(E)$ generated by the projection $p_{v_0}$. Thus we have a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & I_{v_0} \arrow[r, "i"] & C^*(E) \arrow[r, "\pi_E"] & C^*(G) \arrow[r] & 0. \end{tikzcd}$$

We call $E$ an *essential* 1-sink extension of $G$ when $G^0 \geq v_0$. Note that $I_{v_0}$ is an essential ideal of $C^*(E)$ if and only if $E$ is an essential 1-sink extension of $G$ [13, Lemma 2.2].

**Lemma 6.6.** *If $G$ is a row-finite graph and $(E, v_0)$ is an essential 1-sink extension of $G$, then $I_{v_0} \cong \mathcal{K}$.*

*Proof.* Let $E^*(v_0)$ be the set of all paths in $E$ whose range is $v_0$. Since $E$ is an essential 1-sink extension of $G$, it follows that $G^0 \geq v_0$. Thus for every $w \in G^0$ there exists a path from $w$ to $v_0$. If $G^0$ is infinite, this implies that $E^*(v_0)$ is also infinite. If $G^0$ is finite, then because $G^0 \geq v_0$ it follows that $G$ is a finite graph with no sinks, and hence contains a loop. If $w$ is any vertex on this loop, then there is a path from $w$ to $v_0$ and hence $E^*(v_0)$ is infinite. Now because $E^*(v_0)$ is infinite it follows from [9, Corollary 2.2] that $I_{v_0} \cong \mathcal{K}(\ell^2(E^*(v_0))) \cong \mathcal{K}$. ☐

*Definition* 6.7. Let $G$ be a row-finite graph and let $(E, v_0)$ be an essential 1-sink extension of $G$. *The extension associated to $E$* is (the strong equivalence class of) the Busby invariant of any extension

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K} \arrow[r, "i_E"] & C^*(E) \arrow[r, "\pi_E"] & C^*(G) \arrow[r] & 0 \end{tikzcd}$$

where $i_E$ is any isomorphism from $\mathcal{K}$ onto $I_{v_0}$. As with other extensions we shall not distinguish between an extension and its Busby invariant.

*Remark* 6.8. The above extension is well-defined up to strong equivalence. If different choices of $i_E$ are made then it follows from a quick diagram chase that the two associated extensions will be strongly equivalent (see problem 3E(c) of [18] for more details). Also recall that since $p_{v_0}$ is a minimal projection in $I_{v_0}$ [9, Corollary 2.2], it follows that $i_E^{-1}(p_{v_0})$ will always be a rank 1 projection in $\mathcal{K}$.

Let $(E, v_0)$ be a 1-sink extension of $G$. Then for $w \in E^0$ we denote by $Z(w, v_0)$ the set of paths $\alpha$ from $w$ to $v_0$ with the property that $\alpha_i \in E^1 \backslash G^1$ for $1 \leq i \leq |\alpha|$. The *Wojciech vector* of $E$ is the element $\omega_E \in \prod_{G^0} \mathbb{N}$ given by

$$\omega_E(w) := \#Z(w, v_0).$$

An edge $e \in E^1$ with $s(e) \in G^0$ and $r(e) \notin G^0$ is called a *boundary edge*, and the sources of these edges are called *boundary vertices*.

**Lemma 6.9.** *Let $G$ be a row-finite graph and let $(E, v_0)$ be a 1-sink extension of $G$. If $\{s_e, p_v\}$ is the canonical Cuntz-Krieger $E$-family in $C^*(E)$ and $\sigma : C^*(E) \to \mathcal{B}$ is a representation with the property that $\sigma(p_{v_0})$ is a rank 1 projection, then*

$$\operatorname{rank} \sigma(s_e) = \#Z(r(e), v_0) \qquad \textit{for all } e \in E^1 \backslash G^1.$$

*Proof.* For $e \in E^1 \backslash G^1$ let $k_e := \max\{|\alpha| : \alpha \in Z((r(e), v_0)\}$. Since $E$ is a 1-sink extension of $G$ we know that $k_e$ is finite. We shall prove the claim by induction on $k_e$. If $k_e = 0$, then $r(e) = v_0$ and $\operatorname{rank} \sigma(s_e) = \operatorname{rank} \sigma(s_e^* s_e) = \operatorname{rank} \sigma(p_{v_0}) = 1$.

Assume that the claim holds for all $f \in E^1 \backslash G^1$ with $k_f \leq m$. Then let $e \in E^1 \backslash G^1$ with $k_e = m+1$. Since $E$ is a 1-sink extension of $G$ there are no loops based at $r(e)$. Thus $k_f \leq m$ for all $f \in E^1 \backslash G^1$ with $s(f) = r(e)$. By the induction hypothesis $\operatorname{rank} \sigma(s_f) = \# Z(r(e), v_0)$ for all $f$ with $s(f) = r(e)$. Since the projections $s_f s_f^*$ are mutually orthogonal we have

$$\begin{aligned} \operatorname{rank} \sigma(s_e) = \operatorname{rank} \sigma(s_e^* s_e) = \operatorname{rank} \left( \sum_{s(f)=r(e)} \sigma(s_f s_f^*) \right) &= \sum_{s(f)=r(e)} \operatorname{rank} \sigma(s_f s_f^*) \\ = \sum_{s(f)=r(e)} \# Z((r(f), v_0) = \# Z(r(e), v_0). \end{aligned}$$

□

**Lemma 6.10.** *Let $G$ be a row-finite graph with no sinks that satisfies Condition (L), and let $d : \operatorname{Ext}(C^*(G)) \to \operatorname{coker}(B_G - I)$ be the Cuntz-Krieger map. If $(E, v_0)$ is an essential 1-sink extension of $G$ and $\tau$ is the Busby invariant of the extension associated to $E$, then*

$$d(\tau) = [x]$$

*where $[x]$ is the class in $\operatorname{coker}(B_G - I)$ of the vector $x \in \prod_{G^1} \mathbb{Z}$ given by $x(e) := \omega_E(r(e))$ for all $e \in G^1$, and $\omega_E$ is the Wojciech vector of $E$.*

*Proof.* Let $\{s_e, p_v\}$ be the canonical Cuntz-Krieger $G$-family in $C^*(G)$, and let $\{t_e, q_v\}$ be the canonical Cuntz-Krieger $E$-family in $C^*(E)$. Choose an isomorphism $i_E : \mathcal{K} \to I_{v_0}$, and let $\sigma$ and $\tau$ be the homomorphisms that make the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K} \arrow[r, "i_E"] \arrow[d, equal] & C^*(E) \arrow[r, "\pi_E"] \arrow[d, "\sigma"] & C^*(G) \arrow[r] \arrow[d, "\tau"] & 0 \\ 0 \arrow[r] & \mathcal{K} \arrow[r, "i"] & \mathcal{B} \arrow[r, "\pi"] & \mathcal{Q} \arrow[r] & 0 \end{tikzcd}$$

commute. Then $\tau$ is the Busby invariant of the extension associated to $E$, and since $E$ is an essential 1-sink extension, it follows that $\sigma$ and $\tau$ are injective. For all $v \in E^0$ and $e \in E^1$ define

$$H_v := \operatorname{im} \sigma(q_v) \qquad \text{and} \qquad H_e := \operatorname{im} \sigma(t_e t_e^*).$$

Note that $s(e) = v$ implies that $H_e \subseteq H_v$. Also since $i_E^{-1}(q_{v_0})$ is a rank 1 projection, and since the above diagram commutes, it follows that $\sigma(q_{v_0})$ is a rank 1 projection. Thus $H_{v_0}$ is 1-dimensional. Furthermore, by Lemma 6.9 we see that $\dim(H_v) = \# Z(v, v_0)$ and $\dim(H_e) = \# Z(r(e), v_0)$ for all $v \in E^0 \backslash G^0$ and $e \in E^1 \backslash G^1$. In addition, since $t_e t_e^* \leq q_{s(e)}$ for any $e \in E^1 \backslash G^1$ and because the $q_v$'s are mutually orthogonal projections, it follows that the $H_e$'s are mutually orthogonal subspaces for all $e \in E^1 \backslash G^1$.

For all $v \in G^0$ define

$$V_v := H_v \ominus \Big( \bigoplus_{\substack{e \text{ is a boundary} \\ \text{edge and } s(e)=v}} H_e \Big).$$

Then for every $v \in G^0$, we have $\pi(\sigma(q_v)) = \tau(\pi_E(q_v)) = \tau(p_v) \neq 0$ since $\tau$ is injective. Therefore, the rank of $\sigma(q_v)$ is infinite and hence $\dim(H_v) = \infty$ and $\dim(V_v) = \infty$. Now for each $v \in G^0$ and $e \in G^1$ let $P_v$ be the projection onto $V_v$ and $S_e$ be a partial isometry with initial space $V_{r(e)}$ and final space $H_e$. One can then check that $\{S_e, P_v\}$ is a Cuntz-Krieger $G$-family in $\mathcal{B}$. Therefore, by the universal property of $C^*(G)$ there exists a homomorphism $\tilde{t} : C^*(G) \to \mathcal{B}$ with the property that $\tilde{t}(s_e) = S_e$ and $\tilde{t}(p_v) = P_v$. Define $t := \pi \circ \tilde{t}$.

Then for all $v \in G^0$ we have that

$$t(p_v) = \pi(\tilde{t}(p_v)) = \pi(P_v) \neq 0.$$

Thus $p_v \notin \ker t$ for all $v \in G^0$. By Lemma 5.6 it follows that $\ker t = \{0\}$ and $t$ is an essential extension of $C^*(G)$. Now since $S_e S_e^*$ is a projection onto a subspace of $\operatorname{im} \sigma(t_e t_e^*)$ with finite codimension, it follows that $\pi(S_e S_e^*) = \pi(\sigma(t_e t_e^*))$. Thus $t$ has the property that for all $e \in G^1$

$$t(s_e s_e^*) = \pi(\tilde{t}(s_e s_e^*)) = \pi(S_e S_e^*) = \pi(\sigma(t_e t_e^*)) = \tau(\pi_E(t_e t_e^*)) = \tau(s_e s_e^*).$$

By the definition of the Cuntz-Krieger map $d$ it follows that the image of the extension associated to $E$ will be the class of the vector $d_\tau$ in $\operatorname{coker}(B_G - I)$, where $d_\tau(e) = -\operatorname{ind}_{\tau(s_e s_e^*)} \tau(s_e) t(s_e^*)$. Now $\operatorname{ind}_{\tau(s_e s_e^*)} \tau(s_e) t(s_e^*)$ is equal to the Fredholm index of $\sigma(t_e t_e^*) \sigma(t_e) S_e^* \sigma(t_e t_e^*) = \sigma(t_e) S_e^*$ in $\operatorname{im}(\sigma(t_e t_e^*)) = H_e$. Since $S_e$ is a partial isometry with initial space $V_{r(e)} \subseteq H_{r(e)}$ and final space $H_e$, and since $\sigma(t_e)$ is a partial isometry with initial space $H_{r(e)}$ it follows that $\ker \sigma(t_e) S_e^* = \{0\}$ in $H_e$. Furthermore, $\sigma(t_e^*)$ is a partial isometry with initial space $H_e$ and final space

$$H_{r(e)} = V_{r(e)} \oplus \Big( \bigoplus_{\substack{f \text{ is a boundary} \\ \text{edge and } s(f)=r(e)}} H_f \Big)$$

and $S_e$ is a partial isometry with initial space $V_{r(e)}$. Therefore, since $\dim(H_f) = \# Z(r(f), v_0)$ for all $f \notin G^1$ we have that

$$\ker((\sigma(t_e) S_e)^*) = \ker(S_e \sigma(t_e^*)) = \sum_{s(f)=r(e)} Z(r(f), v_0) = \omega_E(r(e)).$$

Thus $d_\tau(e) = \omega_E(r(e))$ for all $e \in G^1$. □

**Proposition 6.11.** *Let $G$ be a row-finite graph with no sinks that satisfies Condition (L), and suppose that $(E, v_0)$ is an essential 1-sink extension of $G$. If $\tau$ is the Busby invariant of the extension associated to $E$, then the value that the Wojciech map $\omega : \operatorname{Ext}(C^*(G)) \to \operatorname{coker}(A_G - I)$ assigns to $\tau$ is given by the class of the Wojciech vector in $\operatorname{coker}(A_G - I)$; that is,*

$$\omega(\tau) = [\omega_E].$$

*Proof.* From Lemma 6.10 we have that $d_\tau = [x]$ in $\operatorname{coker}(B_G - I)$, where $x \in \prod_{G^1} \mathbb{Z}$ is the vector given by $x(e) := \omega_E(r(e))$ for $e \in G^1$. By the definition of $\omega$ we have that $\omega(\tau) := \overline{S_G}(d_\tau)$ in $\operatorname{coker}(A_G - I)$. Thus $\omega(\tau)$ equals the class of the vector $y \in \prod_{G^0} \mathbb{Z}$ given by

$$y(v) = (S_G(x))(v) = \sum_{s(e)=v} x(e) = \sum_{s(e)=v} \omega_E(r(e)).$$

Hence for all $v \in G^0$ we have

$$y(v) - \omega_E(v) = \sum_{s(e)=v} \omega_E(r(e)) - \omega_E(v) = \sum_{w \in G^0} A_G(v, w) \omega_E(w) - \omega_E(v)$$

so $y - \omega_E = (A_G - I)\omega_E$. Thus $[y] = [\omega_E]$ and $\omega(\tau) = [\omega_E]$ in $\operatorname{coker}(A_G - I)$. $\square$

This result gives us a method to prove that $\omega$ is surjective. We need only produce essential 1-sink extensions with the appropriate Wojciech vectors.

A 1-sink extension $E$ of $G$ is said to be *simple* if $E^0 \backslash G^0$ consists of a single vertex. If $G$ is a graph with no sinks, then for any $x \in \prod_{G^0} \mathbb{N}$ we may form a simple 1-sink extension of $G$ with Wojciech vector equal to $x$ merely by defining $E^0 := G^0 \cup \{v_0\}$ and $E^1 := G^1 \cup \{e^i_w : w \in G^0 \text{ and } 1 \leq i \leq x(w)\}$ where each $e^i_w$ is an edge with source $w$ and range $v_0$. In order to show that the Wojciech map is surjective we will not only need to produce such 1-sink extensions, but also ensure that they are essential.

**Lemma 6.12.** *Let $G$ be a row-finite graph with no sinks that satisfies Condition (L). There exists a vector $n \in \prod_{G^0} \mathbb{Z}$ with the following two properties:*

1. $(A_G - I)n \in \prod_{G^0} \mathbb{N}$
2. *for all $v \in G^0$ there exists $w \in G^0$ such that $v \geq w$ and $((A_G - I)n)(w) \geq 1$.*

*Proof.* Let $L \subseteq G^0$ be those vertices of $G$ that feed into a loop; that is,

$$L := \{v \in G^0 : \text{there exists a loop } x \text{ in } G \text{ for which } v \geq r(x_1)\}.$$

Now consider the set $M := G^0 \backslash L$. Because $G$ has no sinks, and because $v \in M$ and $v \geq w$ implies that $w \in M$, it follows that $M$ cannot have a finite number of elements. Thus $M$ is either empty or countably infinite. If $M \neq \emptyset$ then list the elements of $M$ as $M = \{w_1, w_2, \ldots\}$. Now let $v^1_1 := w_1$. Choose an edge $e^1_1 \in G^1$ with the property that $s(e^1_1) = v^1_1$ and define $v^1_2 := r(e^1_1)$. Continue in this fashion: given $v^1_k$ choose an edge $e^1_k$ with $s(e^1_k) = v^1_k$ and define $v^1_{k+1} := r(e^1_k)$. Then $v^1_1, v^1_2, \ldots$ are the vertices of an infinite path which are all elements of $M$. Since these vertices do not feed into a loop it follows that they are distinct; i.e. $v^1_i \neq v^1_j$ when $i \neq j$.

Now if every element $w \in M$ has the property that $w \geq v^1_i$ for some $i$, then we shall stop. If not, choose the smallest $j \in \mathbb{N}$ for which $w_j \not\geq v^1_i$ for all $i \in \mathbb{N}$. Then define $v^2_1 := w_j$ and choose an edge $e^2_1$ with $s(e^2_1) = v^2_1$. Define $v^2_2 := r(e^2_1)$. Continue in this fashion: given $v^2_k$ choose an edge $e^2_k$ with $s(e^2_k) = v^2_k$ and define $v^2_{k+1} := r(e^2_k)$. Then we produce a set of distinct vertices $v^2_1, v^2_2, v^2_3, \ldots$ that lie on the infinite path $e^2_1 e^2_2 e^2_3 \ldots$. Moreover, since $v^2_1 \not\geq v^1_i$ for all $i$ we must have that the $v^2_i$'s are also distinct from the $v^1_i$'s.

Continue in this manner. Having produced an infinite path $e^k_1 e^k_2 e^k_3 \ldots$ with distinct vertices $v^k_1, v^k_2, \ldots$ we stop if every element $w \in M$ has the property that $w \geq v^j_i$ for some $1 \leq i < \infty, 1 \leq j \leq k$. Otherwise, we choose the smallest $l \in \mathbb{N}$ such that $w_l \not\geq v^j_i$ for all $1 \leq i < \infty, 1 \leq j \leq k$. We define $v^{k+1}_1 := w_l$. Given $v^{k+1}_j$ we choose an edge $e^{k+1}_j$ with $s(e^{k+1}_j) = v^{k+1}_j$. We then define $v^{k+1}_{j+1} := r(e^{k+1}_j)$. Thus we produce an infinite path $e^{k+1}_1 e^{k+1}_2 \ldots$ with distinct vertices $v^{k+1}_1, v^{k+1}_2, \ldots$. Moreover, since $v^{k+1}_1 \not\geq v^j_i$ for all $1 \leq i < \infty, 1 \leq j \leq k$, it follows that the $v^{k+1}_i$'s are distinct from the $v^j_i$'s for $j \leq k$.

By continuing this process we are able to produce the following. For some $n \in \mathbb{N} \cup \{\infty\}$ there is a set of distinct vertices $S \subseteq M$ given by

$$S = \{v^k_j : 1 \leq j < \infty, 1 \leq k < n\}$$

with the property that $M \geq S$, and for any $v^k_j \in S$ there exists an edge $e^k_j \in G^1$ for which $s(e^k_j) = v^k_j$ and $r(e^k_j) = v^k_{j+1}$.

Now define

$$a_v = \begin{cases} 1 & \text{if } v \in L \\ j & \text{if } v = v_j^k \in S \\ 0 & \text{otherwise.} \end{cases}$$

and let $n := (a_v) \in \prod_{G^0} \mathbb{Z}$. We shall now show that $n$ has the appropriate properties. We shall first show that $(A_G - I)n \in \prod_{G^0} \mathbb{N}$. Let $v \in G^0$ and consider four cases. (Throughout the following remember that the entries of $n$ are nonnegative integers.)

Case 1: $A_G(v,v) \geq 1$. Then $((A_G - I)n)(v) \geq a_v(A_G(v,v) - 1) \geq 0$.

Case 2: $A_G(v,v) = 0, v \in L$. Since $A_G(v,v) = 0$ and $v$ feeds into a loop, there must exist an edge $e \in G^1$ with $s(e) = v$ and $r(e) \in L$. Thus

$$((A_G - I)n)(v) \geq a_v(A_G(v,v) - 1) + a_{r(e)}A_G(v, r(e)) \geq 1(-1) + 1(1) = 0.$$

Case 3: $A_G(v,v) = 0, v = v_j^k \in S$. Then there exists an edge $e_j^k$ with $s(e_j^k) = v_j^k$ and $r(e_j^k) = v_{j+1}^k \neq v_j^k$. Thus

$$((A_G - I)n)(v) \geq a_v(A_G(v,v) - 1) + a_{v_{j+1}^k}A_G(v, v_{j+1}^k) \geq j(-1) + (j+1)(1) = 1.$$

Case 4: $A_G(v,v) = 0, v \notin L, v \notin S$. Then

$$((A_G - I)n)(v) \geq a_v(A_G(v,v) - 1) \geq 0 \cdot (A_G(v,v) - 1) = 0.$$

Therefore $(A_G - I)n \in \prod_{G^0} \mathbb{N}$.

We shall now show that for all $v \in G^0$ there exists $w \in G^0$ such that $v \geq w$ and $((A_G - I)n)(w) \geq 1$. If $v \notin L$, then $v \in M$ and $v \geq v_j^k$ for some $v_j^k \in S$. But then there is an edge $e_j^k$ with $s(e_j^k) = v_j^k$ and $r(e_j^k) = v_{j+1}^k \neq v_j^k$. Thus we have that

$$\begin{aligned} ((A_G - I)n)(v_j^k) \geq & a_{v_j^k}(A_G(v_j^k, v_j^k) - 1) + a_{v_{j+1}^k}A_G(v_j^k, v_{j+1}^k) \\ \geq & (j)(0 - 1) + (j+1)(1) = 1. \end{aligned}$$

On the other hand, if $v \in L$, then $v$ feeds into a loop. Since $G$ satisfies Condition (L) this loop must have an exit. Therefore, there exists $w \in L$ such that $v \geq w$ and $w$ is the source of two distinct edges $e, f \in G^1$, where one of the edges, say $e$, is the edge of a loop and hence has the property that $r(e) \in L$. Now consider the following three cases.

Case 1: $r(f) \notin L$. Then $r(f) \in M$ and hence $r(f) \geq v_j^k$ for some $v_j^k \in S$. But then $v \geq v_j^k$ and $((A_G - I)n)(v_j^k) \geq 1$ as above.

Case 2: $r(f) \in L$ and $r(e) = r(f)$. Then

$$((A_G - I)n)(w) \geq -a_w + a_{r(f)}A_G(w, r(f)) \geq -1 + (1)(2) = 1.$$

Case 3: $r(f) \in L$ and $r(e) \neq r(f)$. Then

$$\begin{aligned} ((A_G - I)n)(w) \geq & -a_w + a_{r(e)}A_G(w, r(e)) + a_{r(f)}A_G(w, r(f)) \\ \geq & -1 + (1)(1) + (1)(1) = 1. \end{aligned}$$

□

**Lemma 6.13.** *Let $G$ be a row-finite graph with no sinks that satisfies Condition (L). Let $x \in \prod_{G^0} \mathbb{N}$. Then there exists an essential 1-sink extension $E$ of $G$ with the property that $[\omega_E] = [x]$ in* $\operatorname{coker}(A_G - I)$*.*

*Proof.* By Lemma 6.12 we see that there exists $n \in \prod_{G^0} \mathbb{Z}$ with the property that $(A_G - I)n \in \prod_{G^0} \mathbb{N}$ and for all $v \in G^0$ there exists $w \in G^0$ for which $v \geq w$ and $((A_G - I)n)(w) \geq 1$. Since $x + (A_G - I)n \in \prod_{G^0} \mathbb{N}$ we may let $E$ be a 1-sink extension of $G$ with Wojciech vector $\omega_E = x + (A_G - I)n$. Let $v_0$ be the sink of $E$. We shall show that $E$ is essential. Let $v \in G^0$. Then there exists $w \in G^0$ for which $v \geq w$ and $((A_G - I)n) \geq 1$. But then $\omega_E(w) \geq ((A_G - I)n)(w) \geq 1$ and $w$ is a boundary vertex of $E$. Hence $v \geq w \geq v_0$ and we have shown that $G^0 \geq v_0$. Thus $E$ is essential, and furthermore $[\omega_e] = [x + (A_G - I)n] = [x]$ in $\operatorname{coker}(A_G - I)$. ☐

**Proposition 6.14.** *Let G be a row-finite graph with no sinks that satisfies Condition (L). The Wojciech map* $\omega : \operatorname{Ext}(C^*(G)) \to \operatorname{coker}(A_G - I)$ *is surjective.*

*Proof.* If $x$ is any vector in $\prod_{G^0} \mathbb{N}$, then by Lemma 6.13 there exists an essential 1-sink extensions $E$ for which $[\omega_E] = [x]$. If $\tau$ is the Busby invariant of the extension associated to $E$, then by Lemma 6.11 we have that $\omega(\tau) = [\omega_{E_1}] = [x]$. Thus $[x] \in \operatorname{im} \omega$ for all $x \in \prod_{G^0} \mathbb{N}$.

Now because $C^*(G)$ is separable and nuclear, it follows from [3, Corollary 15.8.4] that $\operatorname{Ext}(C^*(G))$ is a group. Because $\prod_{G^0} \mathbb{N}$ is the positive cone of $\prod_{G^0} \mathbb{Z}$, and hence generates $\prod_{G^0} \mathbb{Z}$, the fact that $[x] \in \operatorname{im} \omega$ for all $x \in \prod_{G^0} \mathbb{N}$ implies that $\operatorname{im} \omega = \operatorname{coker}(A_G - I)$. ☐

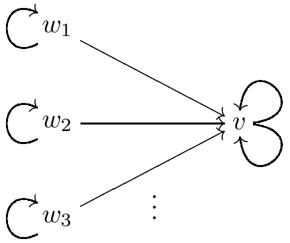
**Corollary 6.15.** *Let G be a row-finite graph with no sinks that satisfies Condition (L). The map* $d : \operatorname{Ext}(C^*(G)) \to \operatorname{coker}(B_G - I)$ *is surjective.*

*Proof.* This follows from the fact that $\omega = \overline{S_G} \circ d$, and $\overline{S_G}$ is an isomorphism. ☐

**Theorem 6.16.** *Let G be a row-finite graph with no sinks that satisfies Condition (L). The Wojciech map* $\omega : \operatorname{Ext}(C^*(G)) \to \operatorname{coker}(A_G - I)$ *and the Cuntz-Krieger map* $d : \operatorname{Ext}(C^*(G)) \to \operatorname{coker}(B_G - I)$ *are isomorphisms. Consequently,*

$$\operatorname{Ext}(C^*(G)) \cong \operatorname{coker}(A_G - I) \cong \operatorname{coker}(B_G - I).$$

*Remark* 6.17. Suppose that $G$ is a row-finite graph with no sinks that satisfies Condition (L), and that $\tau$ is an element of $\operatorname{Ext}(C^*(G))$ for which $\omega(\tau) \in \operatorname{coker}(A_G - I)$ can be written as $[x]$ for some $x \in \prod_{G^0} \mathbb{N}$. Then Lemma 6.13 shows us that there exists an essential 1-sink extension $E$ with the property that the extension associated to $E$ is equal to $\tau$ in $\operatorname{Ext}(C^*(G))$. Thus for every $\tau \in \operatorname{Ext}(C^*(G))$ with the property that $\omega(\tau) = [x]$ for $x \in \prod_{G^0} \mathbb{N}$, we may choose a representative that is the extension associated to an essential 1-sink extension. It is natural to wonder if this is the case for all elements of $\operatorname{Ext}(C^*(G))$. It turns out that in general it is not. To see this let $G$ be the following infinite graph.



Then $G$ is a row-finite graph with no sinks that satisfies Condition (L). However,

$$A_G - I = \begin{pmatrix} 1 & 0 & 0 & \\ 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \\ \vdots & & \ddots & \end{pmatrix},$$

and if we let $x := \begin{pmatrix} -1 \\ -2 \\ -3 \\ \vdots \end{pmatrix} \in \prod_{G^0} \mathbb{Z}$ then for all $n \in \prod_{G^0} \mathbb{Z}$ we have that

$$x + (A_G - I)n = \begin{pmatrix} -1+n(v) \\ -2+n(v) \\ -3+n(v) \\ \vdots \end{pmatrix}.$$

Thus for any $n \in \prod_{G^0} \mathbb{Z}$ we see that $x + (A_G - I)n$ has negative entries. Hence $x+(A_G-I)n$ cannot be the Wojciech vector of a 1-sink extension for any $n \in \prod_{G^0} \mathbb{Z}$.

It turns out, however, that if we add the condition that $G$ be a finite graph then the result does hold.

**Lemma 6.18.** *Let $G$ be a finite graph with no sinks that satisfies Condition (L). If $v \in G^0$, then there exists $n \in \prod_{G^0} \mathbb{N}$ for which $(A_G - I)n \in \prod_{G^0} \mathbb{N}$ and $((A_G - I)n)(v) \geq 1$.*

*Proof.* If $A_G(v, v) \geq 2$ then we can let $n = \delta_v$ and the claim holds. Therefore, we shall suppose that $A_G(v, v) \leq 1$. Since $G$ has no sinks and satisfies Condition (L), there must exist an edge $e_1 \in G^1$ with $s(e_1) = v$ and $r(e_1) \neq v$. Then since $G$ has no sinks we may find an edge $e_2 \in G^1$ with $s(e_2) = r(e_1)$ , and an edge $e_3 \in G^1$ with $s(e_3) = r(e_2)$. Continuing in this fashion we will produce an infinite path $e_1e_2\ldots$ with $s(e_1) = v$. Since $G$ is finite, the vertices $s(e_i)$ of this path must eventually repeat. Let $m$ be the smallest natural number for which $s(e_m) = s(e_k)$ for some $1 \leq k \leq m - 1$. Note that because $r(e_1) \neq s(e_1)$ we must have $m \geq 3$.

Now $e_k e_{k+1} \ldots e_{n-1}$ will be a loop, and since $G$ satisfies Condition (L), there exists an exit for this loop. Thus for some $k \leq l \leq n - 1$ there exists $f \in G^1$ such that $r(f) = s(e_l)$ and $f \neq e_l$. For each $w \in G^0$ define

$$a_w := \begin{cases} 2 & \text{if } w \in \{s(e_i)\}_{i=2}^{l} \\ 1 & \text{otherwise} \end{cases}$$

Note that $\{s(e_i)\}_{i=2}^{l}$ may be empty. This will occur if and only if $l = 1$. Now let $n := (a_w) \in \prod_{G^0} \mathbb{N}$. To see that $((A_G - I)n)(v) \geq 1$, note that $a_v = 1$, and consider four cases.

Case 1: $l = 1$ and $r(f) = r(e_1)$. Since $r(e_1) \neq v$ we have that

$$((A_G - I)n)(v) \geq a_v(A_G(v, v) - 1) + a_{r(e_1)} A_G(v, r(e_1)) \geq 1(-1) + 1(2) = 1.$$

Case 2: $l = 1$ and $r(f) = v$. Then

$$((A_G - I)n)(v) \geq a_v(A_G(v, v) - 1) + a_{r(e_1)} A_G(v, r(e_1)) \geq 1(1 - 1) + 1(1) = 1.$$

Case 3: $l = 1$, $r(f) \neq r(e_1)$, and $r(f) \neq v$. Then

$$\begin{aligned} ((A_G - I)n)(v) &\geq a_v(A_G(v, v) - 1) + a_{r(e_1)} A_G(v, r(e_1)) + a_{r(f)} A_G(v, r(f)) \\ &\geq 1(-1) + 1(1) + 1(1) \\ &= 1. \end{aligned}$$

Case 4: $l \geq 2$. Then $a_{r(e_1)} = 2$ and

$$((A_G - I)n)(v) \geq a_v(A_G(v,v) - 1) + a_{r(e_1)}A_G(v, r(e_1)) \geq 1(-1) + 2(1) = 1.$$

To see that $(A_G - I)n \in \prod_{G^0} \mathbb{N}$ let $w \in G^0$ and consider the following three cases. Case 1: $w = s(e_l)$ and $r(e_l) = r(f)$. Then $a_w = 2$ and we have

$$((A_G - I)n)(w) \geq a_w(A_G(w,w) - 1) + a_{r(e_l)}A_G(w, r(e_l)) \geq 2(-1) + 1(2) = 0.$$

Case 2: $w = s(e_l)$ and $r(e_l) \neq r(f)$. Then

$$\begin{aligned}((A_G - I)n)(w) &\geq a_w(A_G(w,w) - 1) + a_{r(e_l)}A_G(w, r(e_l)) + a_{r(f)}A_G(w, r(f))\\ &\geq 2(-1) + 1(1) + 1(1)\\ &= 0.\end{aligned}$$

Case 3: $w \neq s(e_l)$. Then either $w \in \{s(e_i)\}_{i=2}^{l-1}$ or $a_w = 1$. In either case there exists an edge $e$ with $s(e) = w$ and $a_{r(e)} \geq a_w$. Thus

$$((A_G - I)n)(w) \geq a_w(A_G(w,w) - 1) + a_{r(e)}A_G(w, r(e)) \geq -a_w + a_{r(e)} \geq 0$$

and $(A_G - I)n \in \prod_{G^0} \mathbb{N}$. □

**Theorem 6.19.** *Let $G$ be a finite graph with no sinks that satisfies Condition (L). For any $[x] \in \operatorname{coker}(A_G - I)$ there exists an essential 1-sink extension $E$ of $G$ such that $[\omega_E] = [x]$ in $\operatorname{coker}(A_G - I)$.*

*Proof.* For each $v \in G^0$ we may use Lemma 6.18 to obtain a vector $n_v \in \prod_{G^0} \mathbb{N}$ such that $(A_G - I)n_v \in \prod_{G^0} \mathbb{N}$ and $((A_G - I)n_v)(v) \geq 1$. Now write $x$ in the form $x = \sum_{v \in G^0} a_v \delta_v$. Let $n := \sum_{v \in G^0} (|a_v| + 1)n_v$. Then by linearity, $x + (A_G - I)n \in \prod_{G^0} \mathbb{N}$ and $x + (A_G - I)n \neq 0$. Let $E$ be a 1-sink extension of $G$ with sink $v_0$ and Wojciech vector equal to $x + (A_G - I)n$. Then $[\omega_E] = [x + (A_G - I)n] = [x]$ in $\operatorname{coker}(A_G - I)$. Furthermore, since $\omega_E(v) \geq 1$ for all $v \in G^0$ it follows that $G^0 \geq v_0$ and $E$ is an essential 1-sink extension. □

This result shows that if $G$ is a finite graph with no sinks that satisfies Condition (L), then for any element in $\operatorname{Ext}(C^*(G))$ we may choose a representative that is the extension associated to an essential 1-sink extension $E$ of $G$. Furthermore, since the Wojciech map is an isomorphism we see that if $E_1$ and $E_2$ are essential 1-sink extensions that are representatives for $\tau_1, \tau_2 \in \operatorname{Ext}(C^*(G))$, then the essential 1-sink extension with Wojciech vector equal to $\omega_{E_1} + \omega_{E_2}$ will be a representative of $\tau_1 + \tau_2$. Hence we have a way of choosing representatives of the classes in Ext that have a nice visual interpretation and for which we can easily compute their sum.

## 7. Semiprojectivity of graph algebras

In 1983 Effros and Kaminker [7] began the development of a shape theory for $C^*$-algebras that generalized the topological theory. In their work they looked at $C^*$-algebras with a property that they called semiprojectivity. These semiprojective $C^*$-algebras are the noncommutative analogues of absolute neighborhood retracts. In 1985 Blackadar generalized many of these results [4], but because he wished to apply shape theory to $C^*$-algebras not included in [7] and because the theory in [7] was not a direct noncommutative generalization, Blackadar gave a new definition of semiprojectivity. Blackadar's definition is more restrictive than that in [7].

*Definition* 7.1 (Blackadar). A separable $C^*$-algebra $A$ is *semiprojective* if for any $C^*$-algebra $B$, any increasing sequence $\{J_n\}_{n=1}^\infty$ of (closed two-sided) ideals, and any $*$-homomorphism $\phi : A \to B/J$, where $J := \overline{\bigcup_{n=1}^\infty J_n}$, there is an $n$ and a $*$-homomorphism $\psi : A \to B/J_n$ such that

$$\begin{tikzcd}
A \arrow[r, "\psi"] \arrow[dr, "\phi"'] & B/J_n \arrow[d, "\pi"] \\
 & B/J
\end{tikzcd}$$

where $\pi : B/J_n \to B/J$ is the natural quotient map.

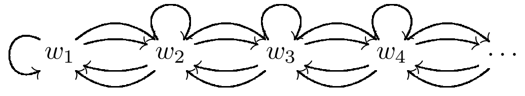
In [4] it was shown that the Cuntz-Krieger algebras are semiprojective, and more recently Blackadar has announced a proof that $\mathcal{O}_\infty$ is semiprojective. Based on the proof for $\mathcal{O}_\infty$ Szymański has proven in [16] that if $E$ is a transitive graph with finitely many vertices (but a possibly infinite number of edges), then $C^*(E)$ is semiprojective.

We now give an example of a row-finite transitive graph $G$ with an infinite number of vertices and with the property that $C^*(G)$ is not semiprojective. We use the fact that the Wojciech map of §6 is an isomorphism in order to prove that $C^*(G)$ is not semiprojective.

If $G$ is a graph, then by *adding a sink at* $v \in G^0$ we shall mean adding a single vertex $v_0$ to $G^0$ and a single edge $e$ to $G^1$ going from $v$ to $v_0$. More formally, if $G$ is a graph, then we form the graph $F$ defined by $F^0 := G^0 \cup \{v_0\}$, $F^1 := G^1 \cup \{e\}$, and we extend $r$ and $s$ to $F^1$ by defining and $r(e) = v_0$ and $s(e) = v$.

*Example* 7.2.

$G$

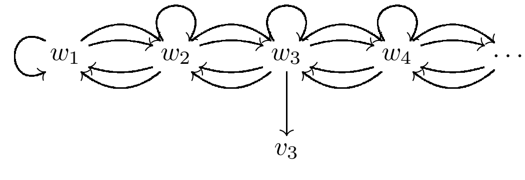


If $G$ is the above graph, then note that $G$ is transitive, row-finite, and has no sinks.

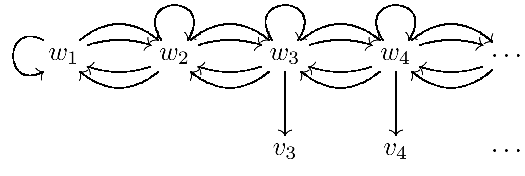
**Theorem 7.3.** *If $G$ is the graph in Example 7.2, then $C^*(G)$ is not semiprojective.*

*Proof.* For each $i \in \mathbb{N}$ let $E_i$ be the graph formed by adding a sink to $G$ at $w_i$, and let $F_i$ be the graph formed by adding a sink to each vertex in $\{w_i, w_{i+1}, \dots\}$. In each case we shall let $v_i$ denote the sink that is added at $w_i$. As examples we draw $E_3$ and $F_3$:

$E_3$



$F_3$



We shall now assume that $C^*(G)$ is semiprojective and arrive at a contradiction. Let $B := C^*(F_1)$ and for each $n \in \mathbb{N}$ let $H_n := \{v_1, v_2, \dots, v_n\}$. Also let $H_\infty :=$

$\{v_1, v_2, \dots\}$. Set $J_n := I_{H_n}$. Then $\{J_n\}_{n=1}^{\infty}$ is an increasing sequence of ideals and $J := \bigcup_{n=1}^{\infty} J_n = I_{H_\infty}$. Now $B/J = C^*(F_1)/I_{H_\infty} \cong C^*(G)$ and for each $n \in \mathbb{N}$, $B/J_n \cong C^*(F_{n+1})$ by [2, Theorem 4.1]. Thus if we identify $C^*(G)$ and $B/J$, then by semiprojectivity there exists a homomorphism $\psi : C^*(G) \to B/J_n$ for some $n$

$$\begin{tikzcd}
C^*(G) \arrow[r, "\psi"] \arrow[dr, "\mathrm{id}"'] & B/J_n \cong C^*(F_{n+1}) \arrow[d, "\pi"] \\
& B/J \cong C^*(G)
\end{tikzcd}$$

such that $\pi \circ \psi = \mathrm{id}$. Note that the projection $\pi : B/J_n \to B/J$ is just the projection $\pi : C^*(F_{n+1}) \to C^*(F_{n+1})/I_{\{v_{n+1}, v_{n+2}, \dots\}} \cong C^*(G)$.

Now if we let $\{s_e, p_v\}$ be the canonical Cuntz-Krieger $F_{n+1}$-family in $C^*(F_{n+1})$ and let $\{t_e, q_v\}$ be the canonical Cuntz-Krieger $E_{n+1}$-family in $C^*(E_{n+1})$, then by the universal property of $C^*(F_{n+1})$ there exists a homomorphism $\rho : C^*(F_{n+1}) \to C^*(E_{n+1})$ such that

$$\rho(s_e) = \begin{cases} t_e & \text{if } e \in E^1_{n+1} \\ 0 & \text{if } e \in F^1_{n+1} \backslash E^1_{n+1} \end{cases} \qquad \text{and} \qquad \rho(p_v) = \begin{cases} q_v & \text{if } v \in E^0_{n+1} \\ 0 & \text{if } v \in F^0_{n+1} \backslash E^0_{n+1}. \end{cases}$$

Since $E_{n+1}$ is a 1-sink extension of $G$, we have the usual projection $\pi_{E_{n+1}} : C^*(E_{n+1}) \to C^*(G)$. One can then check that the diagram

$$\begin{tikzcd}
C^*(F_{n+1}) \arrow[rr, "\rho"] \arrow[dr, "\pi"'] & & C^*(E_{n+1}) \arrow[dl, "\pi_{E_{n+1}}"] \\
& C^*(G) &
\end{tikzcd}$$

commutes simply by checking that $\pi_{E_{n+1}} \circ \rho$ and $\pi$ agree on generators. This, combined with the fact that $\pi \circ \psi = \mathrm{id}$ on $C^*(G)$, implies that $\pi_{E_{n+1}} \circ \rho \circ \psi = \mathrm{id}$. Hence the short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & I_{v_{n+1}} \arrow[r] & C^*(E_{n+1}) \arrow[r, "\pi_{E_{n+1}}"] & C^*(G) \arrow[r] \arrow[l, bend right, "\rho\circ\psi"'] & 0
\end{tikzcd}$$

is split exact. Therefore this extension is degenerate. Since $I_{v_{n+1}} \cong \mathcal{K}$ by [9, Corollary 2.2] we have that this extension is in the zero class in $\mathrm{Ext}(C^*(G))$.

However, the Wojciech vector of $E_{n+1}$ is $\omega_{E_{n+1}} = \delta_{w_{n+1}}$. Since

$$A_G - I = \begin{pmatrix} 0 & 2 & 0 & 0 & \\ 2 & 0 & 2 & 0 & \cdots \\ 0 & 2 & 0 & 2 & \\ 0 & 0 & 2 & 0 & \\ \vdots & & & & \ddots \end{pmatrix}$$

we see that every vector in the image of $A_G - I$ has entries that are multiples of 2. Thus $\delta_{w_{n+1}} \notin \mathrm{im}(A_G - I)$, and $[\omega_{E_{n+1}}]$ is not zero in $\mathrm{coker}(A_G - I)$. But then Proposition 6.11 and Theorem 6.16 imply that the extension associated to $C^*(E_{n+1})$ is not equal to zero in $\mathrm{Ext}(C^*(G))$. This provides the contradiction, and hence $C^*(G)$ cannot be semiprojective. ☐

*Remark* 7.4. After the completion of this work, Spielberg proved in [15] that all classifiable, simple, separable, purely infinite $C^*$-algebras having finitely generated $K$-theory and torsion-free $K_1$-group are semiprojective [15, Theorem 3.12]. This was accomplished by realizing these $C^*$-algebras as graph algebras of transitive graphs. It also implies that if $G$ is a transitive graph that is not a single loop, and if $C^*(G)$ has finitely generated $K$-theory and torsion-free $K_1$-group, then $C^*(G)$ is semiprojective. We mention that the $C^*$-algebra associated to the graph in Example 7.2 does not have finitely generated $K$-theory.

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