

Math 214 Exam 2 — SOLUTIONS
Due Monday, April 10 at the beginning of class

*This exam must be worked on independently. You are allowed to use your notes, the course textbook, and old homework. You are also allowed to talk to the instructor. You are **not** allowed to use any other books, or to talk to anyone else about the exam. Failure to follow these instructions will be considered a violation of the honor code.*

1. (10 points) Let A be a set with a partial order R . Prove that if A is well-ordered, then A is totally ordered.

Proof. Let $x, y \in A$, and let $B = \{x, y\}$. Since A is well-ordered, the set B must have a smallest element. If a is this smallest element, then aRb . If b is this smallest element, then bRa . Thus either aRb or bRa , so R is a total ordering. \square

2. Let A be a set, and let R be a relation on A .

- (a) (5 points) Prove that R is symmetric if and only if $R^{-1} = R$.

Proof. Suppose R is symmetric. Then xRy if and only if yRx . Therefore

$$\begin{aligned}(x, y) \in R &\text{ if and only if } xRy \\ &\text{ if and only if } yRx \\ &\text{ if and only if } (y, x) \in R \\ &\text{ if and only if } (x, y) \in R^{-1}.\end{aligned}$$

Thus $R = R^{-1}$.

Conversely, suppose $R = R^{-1}$. If xRy , then $(x, y) \in R$ and $(y, x) \in R^{-1} = R$. Hence yRx , and R is symmetric. \square

- (b) (5 points) Suppose R is reflexive. Prove that R is transitive if and only if $R \circ R = R$.

Proof. Let R be reflexive, and suppose R is transitive. If $(x, y) \in R$, then since R is reflexive, we have that $(y, y) \in R$. Since $(x, y) \in R$ and $(y, y) \in R$ we have that $(x, y) \in R \circ R$. Thus $R \subseteq R \circ R$. Furthermore, if $(x, y) \in R \circ R$, then there exists $z \in A$ such that $(x, z) \in R$ and $(z, y) \in R$. Thus xRz and zRy , and since R is transitive, it follows that xRy , hence $(x, y) \in R$ and $R \circ R \subseteq R$. Therefore, $R = R \circ R$.

Conversely, let R be reflexive and suppose $R \circ R = R$. If xRy and yRz , then $(x, y) \in R$ and $(y, z) \in R$, so $(x, z) \in R \circ R = R$. Hence xRz , and R is transitive. \square

(Note: This problem shows that a relation R is an equivalence relation if and only if R is reflexive, $R^{-1} = R$, and $R \circ R = R$.)

3. Let $f : A \rightarrow B$ be a function. Define a relation R on A by xRy if and only if $f(x) = f(y)$.

(a) (10 points) Prove that R is an equivalence relation.

Proof. We shall show that R is reflexive, symmetric, and transitive.

Reflexive: If $x \in A$, then $f(x) = f(x)$ so xRx .

Symmetric: If xRy , then $f(x) = f(y)$ and $f(y) = f(x)$. It follows that yRx .

Transitive: If xRy and yRz , then $f(x) = f(y)$ and $f(y) = f(z)$. It follows that $f(x) = f(z)$ and xRz .

□

(b) (10 points) Show that there is a one-to-one correspondence of A/R with $\text{Ran}(f)$.

Proof. Define a function $F : A/R \rightarrow \text{Ran}(f)$ by $F(a/R) = f(a)$. We shall first show that this function is well defined. If $a/R = b/R$, then aRb and by the definition of R we have $f(a) = f(b)$. Thus F is well defined.

Next we shall show that F is surjective. Let $y \in \text{Ran}(f)$. Then $y = f(a)$ for some $a \in A$. Since $a/R \in A/R$ and $F(a/R) = f(a) = y$, we see that F is surjective.

Finally, we shall show that F is injective. Suppose $a/R, b/R \in A/R$ and $F(a/R) = F(b/R)$. Then $f(a) = f(b)$, and consequently aRb . Since R is an equivalence relation, it follows that $a/R = b/R$. Thus F is injective. □

4. (10 points) Consider the following two definitions.

*We say that a partially ordered set A has the **least upper bound property** if every nonempty subset of A that is bounded above has a least upper bound.*

*We say that a partially ordered set A has the **greatest lower bound property** if every nonempty subset of A that is bounded below has a greatest lower bound.*

Prove that if A is a partially ordered set with the least upper bound property, then A has the greatest lower bound property.

Proof. Suppose B is nonempty subset of A that is bounded below. Let $S = \{x \in A : x \text{ is a lower bound of } B\}$. Since B is bounded below, we know that S is nonempty. Furthermore, any element of B is an upper bound for S , and since B is nonempty we know that S is bounded above. Let $a = \sup(S)$. (Since A has the least upper bound property, we know

that $\sup(S)$ exists and $\sup(S) \in A$.) We shall show that a is a greatest lower bound for B .

To begin, let $b \in B$. Because the elements of S are lower bounds for B we must have $x \leq b$ for all $x \in S$. It follows that b is an upper bound for S . Since a is the least upper bound for S , we must have $a \leq b$. Because b was arbitrary, this shows that $a \leq b$ for all $b \in B$. Thus a is a lower bound for B .

Let a_0 be a lower bound for B . Then $a_0 \in S$, and since $a = \sup(S)$, we must have $a_0 \leq a$. Thus a is the greatest lower bound for B . \square

5. (10 points) Let A be a set and let R be a relation on A . We say that R is a **Anti-equivalence Relation** if R has the following three properties:

- (i) Irreflexive: $x \not R x$ for all $x \in A$.
- (ii) Symmetric: If $x R y$, then $y R x$.
- (iii) Antitransitive: If $x R z$, then for any $y \in A$ either $x R y$ or $y R z$.

Prove that if R is an anti-equivalence relation on A , then the complement $S = (A \times A) - R$ is an equivalence relation on A .

(Remark: Just as an equivalence relation is meant to generalize the relation "=", and a partial order is meant to generalize the relation " \leq ", an anti-equivalence relation is meant to generalize the relation " \neq ".)

Proof. We shall show that S is reflexive, symmetric, and transitive.

Reflexive: Let $x \in A$. Then since R is irreflexive we have $x \not R x$, and $(x, x) \notin R$. It follows that $(x, x) \in S$ and $x S x$.

Symmetric: Suppose $x S y$. Then $(x, y) \in S$ and $(x, y) \notin R$. because R is symmetric, it follows that $(y, x) \notin R$. Therefore $(y, x) \in S$ and $y S x$.

Transitive: Suppose $x S y$ and $y S z$. Then $(x, y) \in S$ and $(y, z) \in S$. It follows that $(x, y) \notin R$ and $(y, z) \notin R$, and hence $x \not R y$ and $y \not R z$. By the contrapositive of the antitransitive property we have that $x \not R z$. Hence $(x, z) \notin R$ and $(x, z) \in S$. Thus $x S z$. \square

6. Let $f : A \rightarrow B$ be a function and let $g : B \rightarrow C$ be a function. Answer the following questions by either providing a proof or a counterexample.

- (a) (10 points) If $g \circ f$ is injective, then must it be the case that f is injective?

Answer: Yes.

Proof. Suppose $g \circ f$ is injective. If $x, y \in A$ with $f(x) = f(y)$, then $g(f(x)) = g(f(y))$ and $(g \circ f)(x) = (g \circ f)(y)$. Hence, $x = y$, and f is injective. \square

- (b) (10 points) If $g \circ f$ is injective, then must it be the case that g is injective?

Answer: No

Counterexample: Let $f : [0, \infty) \rightarrow \mathbb{R}$ by $f(x) = \sqrt{x}$, and $g : \mathbb{R} \rightarrow \mathbb{R}$ by $g(x) = x^2$. Then $(g \circ f) : [0, \infty) \rightarrow \mathbb{R}$ by $(g \circ f)(x) = x$, so $g \circ f$ is injective. However, one can easily see that g is not injective (e.g., $g(-3) = g(3)$).

- (c) (10 points) If $g \circ f$ is surjective, then must it be the case that f is surjective?

Answer: No

Counterexample: Let $f : [0, \infty) \rightarrow \mathbb{R}$ by $f(x) = x$, and $g : \mathbb{R} \rightarrow [0, \infty)$ by $g(x) = |x|$. Then $(g \circ f) : [0, \infty) \rightarrow [0, \infty)$ by $(g \circ f)(x) = x$, so $g \circ f$ is surjective. However, one can easily see that f is not surjective because all of its values are non-negative.

- (d) (10 points) If $g \circ f$ is surjective, then must it be the case that g is surjective?

Answer: Yes.

Proof. Suppose $g \circ f$ is surjective. If $c \in C$, then since $g \circ f$ is surjective there exists $a \in A$ such that $(g \circ f)(a) = c$. Let $b = f(a)$. Then

$$g(b) = g(f(a)) = (g \circ f)(a) = c$$

so g is surjective. \square