Math 214 Final Exam — SOLUTIONS
Due Wednesday, May 3 at 8:30AM

This exam must be worked on independently. You are allowed to use your notes, the course textbook, and old homework. You are also allowed to talk to the instructor. You are **not** *allowed to use any other books, or to talk to anyone else about the exam. Failure to follow these instructions will be considered a violation of the honor code.*

1. Identify each of the following statements as either True or False. If the statement is true, provide a proof. If the statement is false, give a counterexample.

   (a) (3 points) There does not exist $m, n \in \mathbb{Z}$ such that $12m + 15n = 2$.
   TRUE.

   Proof. Suppose $m, n \in \mathbb{Z}$ and $12m + 15n = 2$. Then $3(4m + 5n) = 2$ and since $4m + 5n \in \mathbb{Z}$ this implies that 3 divides 2, which is a contradiction. □

   (b) (3 points) There does not exist $m, n \in \mathbb{Z}$ such that $2m + 7n = 8$.
   FALSE.
   Counterexample: Let $m = -3$ and $n = 2$. Then $2m + 7n = 2(-3) + 7(2) = 8$.

   (c) (3 points) For every $a \in \mathbb{Z}$ there exists $b \in \mathbb{Z}$ such that $ab < a$.
   FALSE.
   Counterexample: Let $a = 0$. Then for every $b \in \mathbb{Z}$ we have $ab = 0$ and $ab = a$.

2. (6 points) Let $a, b, c \in \mathbb{Z}$. Prove that if a divides $b - 5$ and a divides $c - 5$, then a divides $bc - 25$.

   Proof. Since a divides $b - 5$, and a divides $c - 5$, it follows that a divides $(b - 5)(c - 5) + 5(b - 5) + 5(c - 5)$. But

   $$(b-5)(c-5)+5(b-5)+5(c-5) = bc-5b-5c+25+5b-25+5c-25 = bc-25.$$

   □

3. (10 points) Prove that if $n \in \mathbb{N}$, then 8 divides $5^{2n} - 1$.

   Proof. We will prove this by induction on n.

   Base Case: $n = 1$

   We have that $5^(2(1)) - 1 = 25 - 1 = 24$, which is divisible by 8 so the base case holds.

Inductive Step: Assume the claim is true for n.

We shall prove the claim holds for $n+1$. By hypothesis, we may write $5^{2n} - 1 = 8k$ for some $k \in \mathbb{Z}$. Then

$$\begin{aligned} 5^{2(n+1)} - 1 &= 5^{2n+2} - 1 = (25)5^{2n} - 1 = 25(5^{2n} - 1) + 24 \\ &= 25(8k) + 24 = 8(25k + 3) \end{aligned}$$

and since $25k + 3 \in \mathbb{Z}$, we have that 8 divides $5^{2(n+1)}$, and the claim holds for $n+1$.

By the Principle of Mathematical Induction, the claim holds for all $n \in \mathbb{N}$. □

4. (10 points) Let A, B, and C be sets. Prove that if $A \subseteq B \cup C$ and $A \cap B = \emptyset$, then $A \subseteq C$.

Proof. Let $x \in A$. Since $A \subseteq B \cup C$, it follows that $x \in B \cup C$. Therefore $x \in B$ or $x \in C$. However, x cannot be in B since $x \in A$ and $A \cap B = \emptyset$. Therefore $x \in C$, and we have shown that $A \subseteq C$. □

5. (10 points) Let A be a set, and suppose that $f : A \to A$, $g : A \to A$, and $h : A \to A$ are functions. Prove that if $f \circ g$ and $g \circ h$ are bijections, then f, g and h are bijections.

Proof. Since $f \circ g$ and $g \circ h$ are each injective, it follows that $f \circ g$ and $g \circ h$ are each invertible. Thus $(f \circ g)^{-1}$ exists and $(g \circ h)^{-1}$ exists. Furthermore, we see that

$$[(f \circ g)^{-1} \circ f] \circ g = (f \circ g)^{-1} \circ (f \circ g) = I_A$$

and therefore $(f \circ g)^{-1} \circ f$ is a left inverse for g. Since g has a left inverse, it follows that g is injective. In addition

$$g \circ [h \circ (g \circ h)^{-1}] = (g \circ h) \circ (g \circ h)^{-1} = I_A$$

and $h \circ (g \circ h)^{-1}$ is a right inverse for g. Because g has a right inverse, it follows that g is surjective. Thus g is a bijection.

Also, since g is a bijection, we know that g is invertible so that g^{-1} exists and g^{-1} is a bijection [Theorem 4.12(a)]. We then have that

$$f = (f \circ g) \circ g^{-1},$$

and since f is the composition of bijective functions, f is bijective [Theorem 4.12(b)]. Likewise,

$$h = g^{-1} \circ (g \circ h)$$

and since h is the composition of bijective functions, h is bijective [Theorem 4.12(b)]. □

6. (10 points) Let $f : A \to B$ and $K \subseteq B$. Prove that $f(f^{-1}(K)) = K \cap \text{Ran}(f)$.

Proof. Let $x \in f(f^{-1}(K))$. Then $x = f(a)$ for some $a \in f^{-1}(K)$. Hence $x \in \text{Ran}(f)$. Furthermore, since $a \in f^{-1}(K)$, we know that $f(a) \in K$ and $x \in K$. Thus $x \in K \cap \text{Ran}(f)$, and $f(f^{-1}(K)) \subseteq K \cap \text{Ran}(f)$.

Conversely, suppose $x \in K \cap \text{Ran}(f)$. Since $x \in \text{Ran}(f)$ we know $x = f(a)$ for some $a \in A$. But because $f(a) = x \in K$, it is the case that $a \in f^{-1}(K)$. Hence $f(a) \in f(f^{-1}(K))$, and $x \in f(f^{-1}(K))$. Thus $f(f^{-1}(K)) = K \cap \text{Ran}(f)$. □

7. If A and B are subsets of $\mathbb{R}$ define

$$A + B = \{a + b : a \in A \text{ and } b \in B\}.$$

(a) (4 points) Prove that if A and B are nonempty subsets of $\mathbb{R}$ that are bounded above, then $A + B$ is bounded above.

Proof. Let M be an upper bound for A and N be an upper bound for B. Then $a \leq M$ for all $a \in A$, and $b \leq N$ for all $b \in B$. It follows that $a + b \leq M + N$ for all $a \in A$ and $b \in B$. Therefore $x \leq M + N$ for all $x \in A + B$. Hence $M + N$ is an upper bound for the set $A + B$. □

(b) (6 points) Prove that if A and B are nonempty subsets of $\mathbb{R}$ that are bounded above, then $\sup(A + B) = \sup(A) + \sup(B)$.

Proof. Since A and B are nonempty and bounded above, and because $\mathbb{R}$ has the least upper bound property, we know that $\sup(A)$ and $\sup(B)$ exist.

We shall first show that $\sup(A) + \sup(B)$ is an upper bound of $A + B$. Since $a \leq \sup(A)$ for all $a \in A$, and $b \leq \sup(B)$ for all $b \in B$. It follows that $a + b \leq \sup(A) + \sup(B)$ for all $a \in A$ and $b \in B$. Therefore $x \leq \sup(A) + \sup(B)$ for all $x \in A + B$. Hence $\sup(A) + \sup(B)$ is an upper bound for the set $A + B$.

Next we shall show that $\sup(A) + \sup(B)$ is the smallest upper bound of $A + B$. Let N be any upper bound for $A + B$.

If $b \in B$, then

$$a + b \leq N \qquad \text{for all } a \in A$$

and

$$a \leq N - b \qquad \text{for all } a \in A.$$

Hence $N - b$ is an upper bound for A, and $\sup(A) \leq N - b$, or equivalently $b \leq N - \sup(A)$. Furthermore, since b is was an arbitrary element in B we may conclude that

$$b \leq N - \sup(A) \qquad \text{for all } b \in B.$$

Hence $N - \sup(A)$ is an upper bound for B, and $\sup(B) \le N - \sup(A)$, or equivalently $\sup(A) + \sup(B) \le N$. Therefore $\sup(A) + \sup(B)$ is the smallest upper bound of $A + B$, and $\sup(A + B) = \sup(A) + \sup(B)$. □

8. Define a relation R on $\mathbb{R} \times (\mathbb{R} - \{0\})$ by $(x, y)R(z, w)$ if and only if $xw = yz$.

   (a) (3 points) Prove that R is an equivalence relation.

   Proof. We shall show that R is reflexive, symmetric, and transitive.
   Reflexive: If $(x, y) \in \mathbb{R} \times (\mathbb{R} - \{0\})$, then $xy = xy$, so $(x, y)R(x, y)$.
   Symmetric: If $(x, y)R(z, w)$, then $xw = yz$, and $zy = wx$ so $(z, w)R(x, y)$.
   Transitive: If $(x, y)R(z, w)$ and If $(z, w)R(s, t)$, then $xw = yz$ and $zt = ws$. Multiplying each side of the second equation by y gives $yzt = yws$. Substituting $xw = yz$ into this equation yields $xwt = yws$, and since $w \neq 0$ we may cancel to obtain $xt = ys$. Thus $(x, y)R(s, t)$. □

   (b) (1 point) Give a geometric description of the equivalence classes of R.

   Answer: $(x, y)/R$ is the set of points on the line through the origin and the point (x, y).

   (c) (3 points) Let S denote the collection of equivalence classes of the relation R. Define a function $F : S \times S \to S$ by

$$F\big((x, y)/R, (z, w)/R\big) = (xw + zy, yw)/R.$$

   Prove that the function A is well-defined.

   Proof. If $\big((x, y)/R, (z, w)/R\big) = \big((x', y')/R, (z', w')/R\big)$, then we have $(x, y)/R = (x', y')/R$ and $(z, w)/R = (z', w')/R$. It follows that $(x, y)R(x', y')$ and $(z, w)R(z', w')$, so that $xy' = yx'$ and $zw' = wz'$. Because $y' \neq 0$ and $w' \neq 0$ we have

$$\begin{aligned} \frac{xw + zy}{yw} &= \frac{(xw + zy)(y'w')}{(yw)(y'w')} \\ &= \frac{xy'ww' + zw'yy'}{ywy'w'} \\ &= \frac{yx'ww' + wz'yy'}{ywy'w'} \\ &= \frac{(x'w' + z'y')(yw)}{ywy'w'} \\ &= \frac{x'w' + z'y'}{y'w'} \end{aligned}$$

   so $F\big((x, y)/R, (z, w)/R\big) = F\big((x', y')/R, (z', w')/R\big)$ and F is well defined. □

(d) (2 points) Where have you encountered this relation in context of elementary arithmetic? What does the function F correspond to?

Answer: This equivalence relation is used to describe when fractions are equal. The function F is the operation of addition on fractions.

(e) (1 point) Could we use the same rule to define an equivalence relation on $\mathbb{R}\times\mathbb{R}$ — that is, is the relation S defined on $\mathbb{R}\times\mathbb{R}$ by $(x, y)S(z, w)$ if and only if $xw = yz$ an equivalence relation?

Answer: No. It is not transitive. We see that $(1, 2)S(0, 0)$ and $(0, 0)S(1, 1)$, but $(1, 2)\not S(1, 1)$.

9. (10 points) Prove that if A is a denumerable set and F is a finite set, then the set $A - F$ is denumerable.

Proof. We see that $A \subseteq (A - F) \cup F$. Since any subset of a finite set is finite [Theorem 5.6], and since A is not finite, it follows that $(A - F) \cup F$ is not finite. Also, since the union of finite sets is finite, and because F is finite and $(A - F) \cup F$ is not finite, we must have that $A - F$ is not finite. Thus $A - F$ is infinite.

In addition, since $A - F \subseteq A$ and A is countable, it follows from Theorem 5.23 that $A - F$ is countable. Since $A - F$ is countable and infinite, $A - F$ is denumerable. □

10. (15 points) For each of the following sets, identify whether the set is **finite**, **denumerable**, or **uncountable**. Explain why your answer is correct. (Hint: In some parts it may help to remember that a function $f : \mathbb{N} \to A$ may be thought of as a sequence of elements in A.)

(a) The set of functions from $\{a, b, c\}$ to $\{1, 2\}$.

Answer: FINITE. For each function there are 2 choices for the value assigned to a, 2 choices for the value assigned to b, and 2 choices for the value assigned to c. Thus there are $2^3 = 8$ functions in this set.

(b) The set of of functions from $\{a, b, c\}$ to $\mathbb{N}$.

Answer: DENUMERABLE. The function that sends f to the ordered triple $(f(a), f(b), f(c))$ is a bijection between the set of these functions and $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$, which is denumerable.

(c) The set of functions from $\mathbb{N}$ to $\{a, b, c\}$

Answer: UNCOUNTABLE. The set is clearly infinite. Since any such function may be thought of as a sequence whose terms are in $\{a, b, c\}$, an argument similar to the one in Theorem 5.14 shows that these sequences cannot be put in a list; that is, they are not countable.

(d) The set of functions from $\mathbb{N}$ to $\mathbb{N}$ that are *eventually one*. (Note: a function $f : \mathbb{N} \to \mathbb{N}$ is said to be eventually one if there exists $N \in \mathbb{N}$ such that $f(n) = 1$ for all $n \geq N$.

Answer: DENUMERABLE. Let $S_N = \{f : \mathbb{N} \to \mathbb{N} : f(n) = 1 \text{ for all } n \geq N+1\}$. Then the function $F : S_N \to \underbrace{\mathbb{N} \times \ldots \mathbb{N}}_{N \text{ times}}$ defined by $F(f) = (f(1), f(2), \ldots, f(N))$ is a bijection. Hence S_N is denumerable. But the set of functions from $\mathbb{N}$ to $\mathbb{N}$ that are eventually one is equal to the union $\bigcup_{N=1}^{\infty} S_N$. Since this is the denumerable union of denumerable sets, it is denumerable by Theorem 5.28.

(e) The set of functions from $\mathbb{N}$ to $\mathbb{N}$ that are *eventually constant*. (Note: a function $f : \mathbb{N} \to \mathbb{N}$ is said to be eventually constant if there exists $c \in \mathbb{N}$ and $N \in \mathbb{N}$ such that $f(n) = c$ for all $n \geq N$.

Answer: DENUMERABLE. For $c \in \mathbb{N}$ let

$$X_c = \{f : \mathbb{N} \to \mathbb{N} : f \text{ is eventually constant with a constant value of } c\}.$$

Then an argument similar to that in Part(d) shows that X_c is denumerable. We see that the set of functions from $\mathbb{N}$ to $\mathbb{N}$ that are eventually constant is equal to the union $\bigcup_{c=1}^{\infty} X_c$. Since this is the denumerable union of denumerable sets, it is denumerable by Theorem 5.28.