**Math 214 Exam 1 — SOLUTIONS**
**Due Friday, Feb. 24 at the beginning of class**

*This exam must be worked on independently. You are allowed to use your notes, the course textbook, and old homework. You are also allowed to talk to the instructor. You are* **not** *allowed to use any other books, or to talk to anyone else about the exam. Failure to follow these instructions will be considered a violation of the honor code.*

1. (10 points) Let $m, n \in \mathbb{Z}$. Prove that if $mn$ is odd, then $m + n$ is even.

   *Proof.* Suppose $mn$ is odd. Since the product of an even number with any other integer is even, it must be the case that both $m$ and $n$ are odd. Thus $m = 2k + 1$ and $n = 2j + 1$ for some $k, j \in \mathbb{Z}$. It follows that $m + n = 2k + 1 + 2j + 1 = 2(k + j + 1)$ and since $k + j + 1 \in \mathbb{Z}$, we have that $m + n$ is even. □

2. (12 points) We all know that $3^2 + 4^2 = 5^2$. Prove that there do not exist three consecutive natural numbers such that the cube of the largest is equal to the sum of the cubes of the other two.

   *Proof.* Suppose that there are consecutive number $k$, $k+1$, $k+2$ for $k \in \mathbb{N}$ with the property that $k^3 + (k + 1)^3 = (k + 2)^3$. Then

   $$k^3 + k^3 + 3k^2 + 3k + 1 = k^3 + 6k^2 + 12k + 8$$

   and

   $$k^3 - 3k^2 - 9k = 7$$

   and

   $$k(k^2 - 3k - 9) = 7.$$

   Since $k^2 - 3k - 9 \in \mathbb{Z}$, this implies that $k \mid 7$. Because 7 is prime and $k$ is a natural number, it follows that $k = 1$ or $k = 7$. However, one can check that neither 1 nor 7 satisfies the equation $k^3 + (k + 1)^3 = (k + 2)^3$. Hence we have a contradiction. □

3. (12 points) Prove that if $a, b, c \in \mathbb{Z}$ with $a^2 + b^2 = c^2$, then either $a$ is even or $b$ is even.

   *Proof.* Suppose $a^2 + b^2 = c^2$, and for the sake of contradiction suppose that $a$ and $b$ are both odd. Then $a = 2k + 1$ and $b = 2j + 1$ for some $k, j \in \mathbb{Z}$. Hence

   $$(2k + 1)^2 + (2j + 1)^2 = c^2$$

   and

   $$4k^2 + 4k + 1 + 4j^2 + 4j + 1 = c^2$$

and

$$4(k^2 + j^2 + k + j) + 2 = c^2. \tag{1}$$

It follows that $2[2(k^2 + j^2 + k + j) + 1] = c^2$, and since $2(k^2 + j^2 + k + j) + 1 \in \mathbb{Z}$ this implies that $c^2$ is even. It then follows from [Problem 3(a), Assignment 3] that $c$ is even. Therefore, $c = 2l$ for some $l \in \mathbb{Z}$, and we have from Eq.1 that

$$4(k^2 + j^2 + k + j) + 2 = 4l^2$$

and

$$2 = 4(l^2 - k^2 - j^2 - k - j).$$

Because $l^2 - k^2 - j^2 - k - j \in \mathbb{Z}$ this implies that $4 \mid 2$, which is a contradiction. □

4. (12 points) Let $n$ be a natural number. Prove that either $n$ is prime or $n$ is a perfect square or $n$ divides $(n-1)!$.

   *Proof.* Suppose that $n$ is neither prime nor a perfect square. Since $n$ is not prime, we may factor $n$ as $n = ab$ where $1 < a < n$ and $1 < b < n$. Also, since $n$ is not a perfect square we know that $a \neq b$. Without loss of generality, we will assume $a > b$. Thus

$$\begin{aligned}(n-1)! &= (n-1)(n-2)\dots a \dots b \dots 2 \cdot 1 \\ &= [(n-1)(n-2)\dots(a+1)(a-1)\dots(b+1)(b-1)\dots 2 \cdot 1](ab) \\ &= [(n-1)(n-2)\dots(a+1)(a-1)\dots(b+1)(b-1)\dots 2 \cdot 1]\ n\end{aligned}$$

   so that $n \mid (n-1)!$. □

5. (10 points) A sequence $\{a_k\}_{k=1}^{\infty}$ is said to be a Cauchy sequence if for every real number $\epsilon > 0$ there exists a natural number $N$ such that $|a_n - a_m| < \epsilon$ whenever $m, n > N$. Give a useful description of what it means for the sequence $\{a_k\}_{k=1}^{\infty}$ to *not* be Cauchy.

   **Answer:** A sequence $\{a_k\}_{k=1}^{\infty}$ is not a Cauchy sequence if for every real number $\epsilon > 0$ and for every natural number $N$, there exists natural numbers $m, n > N$ for which $|a_n - a_m| \geq \epsilon$.

6. (12 points) Prove or find a counterexample to the following statement: For all $n \in \mathbb{N}$ one has $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} \geq \sqrt{n}$.

   *Proof.* This will be a proof by induction on $n$.
   Base Case: Let $n = 1$. Then $\frac{1}{\sqrt{1}} = \sqrt{1}$, so the claim holds for $n = 1$.

Inductive Step: Assume the claim holds for $n$. Then

$$\begin{aligned}
\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} &\geq \sqrt{n} + \frac{1}{\sqrt{n+1}} \\
&= \frac{\sqrt{n}\sqrt{n+1}+1}{\sqrt{n+1}} \\
&\geq \frac{\sqrt{n}\sqrt{n}+1}{\sqrt{n+1}} \\
&= \frac{n+1}{\sqrt{n+1}} \\
&= \sqrt{n+1}
\end{aligned}$$

so the claim holds for $n+1$.

By the Principle of Mathematical Induction, we may conclude that the claim is true for all $n \in \mathbb{N}$. □

7. (10 points) Prove or find a counterexample to the following statement: Let $n \in \mathbb{N}$. If $n^2 - 1$ is divisible by 5, then $n$ is divisible by 2 or 3.

   **Counterexample:** Let $n = 11$. Then $n^2 - 1 = 120$ is divisible by 5, but neither 2 nor 3 divides $n = 11$.

8. (12 points) Let $A$ and $B$ be sets. Prove that $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

   *Proof.* Let $x \in (A - B) \cup (B - A)$. Then $x \in A - B$ or $x \in B - A$. Thus either $x \in A$ and $x \notin B$, or $x \notin A$ and $x \in B$. In the first case, $x \in A$ and $x \notin B$ implies $x \in A \cup B$ and $x \notin A \cap B$. Thus $x \in (A \cup B) - (A \cap B)$. In the second case, $x \notin A$ and $x \in B$ implies $x \in A \cup B$ and $x \notin A \cap B$. Thus $x \in (A \cup B) - (A \cap B)$. Therefore in either case we have $x \in (A \cup B) - (A \cap B)$, so $(A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B)$.

   Let $x \in (A \cup B) - (A \cap B)$. Then $x \in (A \cup B)$ and $x \notin (A \cap B)$. Thus $x \in A$ or $x \in B$, but $x$ is not in both $A$ and $B$. Therefore, either $x \in A - B$ or $x \in B - A$, and $x \in (A - B) \cup (B - A)$. Consequently $(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A)$, and it follows that $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$. □

9. (10 points) Let $A$, $B$, $C$, and $D$ be sets. Prove that if $C \subseteq A$ and $D \subseteq B$, then $D - A \subseteq B - C$.

   *Proof.* Assume that $C \subseteq A$ and $D \subseteq B$, and let $x \in D - A$. Then $x \in D$ and $x \notin A$. Since $D \subseteq B$ and $x \in D$, it follows that $x \in B$. Also, since $C \subseteq A$ and $x \notin A$, it follows that $x \notin C$. Therefore $x \in B - C$, and $D - A \subseteq B - C$. □