Assignment 5

The following problems are due at the beginning of class on Friday, Mar. 3.

**Problem 1:** For \( n \in \mathbb{N} \) let \( P(n) \) denote the proposition “\( n^2 + 5n + 1 \) is even.”

(a) Prove that if \( P(n) \) is true, then \( P(n + 1) \) is true.

(b) For which \( n \) is \( P(n) \) actually true? What is the moral of this exercise?

**Problem 2:** What is wrong with the following proof using induction?

**Theorem:** All horses have the same color.

**Proof:** We establish this well-known fact by mathematical induction. Clearly, all horses in any set of 1 horse have the same color. This completes the base step of the induction. Now assume that all horses in any set of \( n \) horses have the same color. Consider a set of \( n + 1 \) horses, labeled with the integers \( 1, 2, \ldots, n + 1 \). By the induction hypothesis, the horses \( 1, 2, \ldots, n \) all have the same color, as do the horses \( 2, 3, \ldots, n + 1 \). Since the two sets have common members, namely \( 2, 3, \ldots, n \), all \( n + 1 \) horses must have the same color.

What is the moral of this exercise?

**Problem 3:** Prove that

\[
1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}
\]

for all \( n \geq 1 \).

**Problem 4:** Let \( \{f_n\}_{n=1}^{\infty} \) denote the Fibonacci sequence defined in class (and on p.113 of the book). Define the “Fibonacci-2” numbers \( g_n \) by

\[
g_1 = 2, \ g_2 = 2, \text{ and } g_{n+2} = g_{n+1}g_n \quad \text{for all } n \in \mathbb{N}.
\]

(a) Calculate the first five “Fibonacci-2” numbers.

(b) Show that \( g_n = 2^{f_n} \) for all \( n \in \mathbb{N} \).