Exam 1

The following are due at the beginning of class on Friday, Sept. 26.

Problem 1: Let $X$ be a vector space, and suppose that $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on $X$ and that $\mathcal{T}_1$ and $\mathcal{T}_2$ are the corresponding topologies determined by these norms. Prove that if $X$ is complete in both norms and $\mathcal{T}_1 \subseteq \mathcal{T}_2$, then $\mathcal{T}_1 = \mathcal{T}_2$.

Problem 2: Let $X$ be a Hilbert space, and let $E$ be an orthonormal basis for $X$. Prove that if $\{h_n\}_{n=1}^{\infty}$ is a sequence of vectors in $X$, then $\lim_{n \to \infty} \langle h_n, x \rangle = 0$ for all $x \in X$ if and only if $\sup\{\|h_n\| : n \in \mathbb{N}\} < \infty$ and $\lim_{n \to \infty} \langle h_n, e \rangle = 0$ for all $e \in E$.

Problem 3: Let $X$ be a normed space, let $\{x_1, \ldots, x_n\}$ be a linearly independent set of vectors in $X$, and let $z_1, \ldots, z_n \in \mathbb{C}$. Prove that there exists a bounded linear functional $f : X \to \mathbb{C}$ such that $f(x_i) = z_i$ for all $1 \leq i \leq n$.

Problem 4: Let $X$ and $Y$ be Banach spaces, and suppose that $T : X \to Y$ is a bounded linear map. Prove that there exists $c > 0$ such that $\|T(x)\| \geq c\|x\|$ for all $x \in X$ if and only if $\ker T = \{0\}$ and $\text{Ran} T$ is closed.

Problem 5: Let $M$ be the following subspace of the Banach space $\ell^\infty$:

$$M := \{(x_1, x_2, \ldots) \in \ell^\infty : x_i = 0 \text{ for all but finitely many } i\}.$$ 

What is the closure of $M$? Prove that there exists a bounded linear functional $f : \ell^\infty \to \mathbb{C}$ such that $f(x) = 0$ for all $x \in M$ and $f(1,1,1,\ldots) = 1$. 