Exam 2

The following are due at the beginning of class on Friday, October 31.

Problem 1: (20 points) Let $H$ be a Hilbert space. Prove that if $T \in K(H)$, then either rank $T < \infty$ or Ran($T$) is not closed.

Problem 2: If $V$ is a vector space over $\mathbb{C}$, then a vector space partial order on $V$ is a relation $\leq$ satisfying the following:

(i) $x \leq x$ for all $x \in V$.
(ii) If $x, y \in V$ with $x \leq y$ and $y \leq x$, then $x = y$.
(iii) If $x, y, z \in V$ with $x \leq y$ and $y \leq z$, then $x \leq z$.
(iv) If $x, y \in V$ and $x \leq y$, then $x + z \leq y + z$ for all $z \in V$.
(v) If $x, y \in V$ and $x \leq y$, then $rx \leq ry$ for all $r \in [0, \infty)$.

Note that properties (i)–(ii) simply say that $\leq$ is a partial order, and properties (iv)–(v) require that partial order to interact appropriately with the addition and scalar multiplication on $V$.

If $V$ is a vector space over $\mathbb{C}$, a cone in $V$ is a subset $C \subseteq V$ satisfying:

(I) If $x, y \in C$, then $x + y \in C$.
(II) If $x \in C$ and $r \in [0, \infty)$, then $rx \in C$.
(III) $C \cap -C = \{0\}$.

(a) (10 points) Prove that if $\leq$ is a vector space partial order, then $C := \{x \in V : x \geq 0\}$ is a cone.

(b) (10 points) Prove that if $C$ is a cone, and if we define a relation $\leq$ on $V$ by $x \leq y$ if and only if $y - x \in C$, then $\leq$ is a vector space partial order on $V$.

(c) (20 points) Let $H$ be a Hilbert space. Prove that the positive operators in $B(H)$ form a cone. Conclude that we may define a vector space partial ordering on $B(H)$ by: $S \leq T$ if and only if $T - S$ is a positive operator.
(d) (10 points) We already defined a partial ordering \( \leq_p \) on the projections as follows: If \( P, Q \) are projections on \( H \), then \( P \leq_p Q \) if and only if \( QP = P \). Prove that this partial ordering on the projections coincides with the partial ordering provided by the positive operators in \( B(H) \); that is, prove that if \( Q \) and \( P \) are projections, then \( P \leq_p Q \) if and only if \( P \leq Q \). (Feel free to use any results we proved in class.)

**Problem 3:** If \( T : H \to H \) is linear (but not necessarily bounded), we say that \( T \) is **orthogonally diagonalizable** if there exists an orthonormal basis \( \{ e_i \}_{i \in I} \) such that for each \( i \in I \) we have \( T(e_i) = \lambda_i e_i \) for some \( \lambda_i \in \mathbb{C} \).

Suppose that \( H \) is a separable infinite-dimensional Hilbert space, that \( T \) is orthogonally diagonalizable, and that \( \{ e_i \}_{i=1}^{\infty} \) is a countably infinite orthonormal basis for \( H \) with \( T(e_i) = \lambda_i e_i \) for \( i \in \mathbb{N} \). Note that with this choice of basis, we may identify \( H \) with \( \ell^2(\mathbb{N}) \) and we may identify \( T \) with the diagonal infinite matrix indexed by \( \mathbb{N} \) whose diagonal entries are \( \lambda_1, \lambda_2, \ldots \).

(a) (10 points) Prove that \( T \) is a bounded operator if and only if \( \lambda_1, \lambda_2, \ldots \) is a bounded sequence.

(b) (10 points) Prove that \( T \) is a compact operator if and only if the sequence \( \lambda_1, \lambda_2, \ldots \) has the property that \( \lim_{n \to \infty} \lambda_n = 0 \).

(c) (10 points) Prove that \( T \) is a finite-rank operator if and only if the sequence \( \lambda_1, \lambda_2, \ldots \) has only a finite number of nonzero terms.

Note that if we identify the orthogonally diagonalizable on \( H \) with the sequence \( \{ \lambda_n \}_{n=1}^{\infty} \) of eigenvalues (equivalently, diagonal entries), then the bounded operators correspond to \( \ell^\infty := L^\infty(\mathbb{N}) \), the compact operators correspond to \( c_0 := C_0(\mathbb{N}) \), and the finite-rank operators correspond to \( c_{00} := C_c(\mathbb{N}) \).