Assignment 1

The following problems are due at the beginning of class on Friday, Sept. 19.

**Problem 1:** Let \((X, \| \cdot \|)\) be a normed space. Prove that \(\| \cdot \|\) is the norm induced from an inner product if and only if \(\| \cdot \|\) satisfies the *parallelogram law*:

\[
\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \text{for all } x, y \in X.
\]

**Problem 2:** Let \((X, \langle \cdot, \cdot \rangle)\) be a Hilbert space. Prove that the closed unit ball of \(X\) is compact if and only if \(X\) is finite dimensional.

**Problem 3:** Let \(X\) be a vector space. Prove that all norms on \(X\) are equivalent if and only if \(X\) is finite dimensional.

**Problem 4:** Recall that a topological space is defined to be *separable* if it has a countable dense subset. Prove that a Hilbert space \((X, \langle \cdot, \cdot \rangle)\) is separable if and only if it has a countable orthonormal basis.

**Problem 5:** Prove that if \(X\) is an infinite-dimensional Banach space, then any vector space basis of \(X\) is uncountable. (Hint: Use the Baire category theorem.) (This result shows that any Banach space has vector space dimension that is either finite or uncountable.)