DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **14**, Number **2**, February **2006**

Website: http://aimSciences.org

pp. **355–363**

A NOTE ABOUT STABLE TRANSITIVITY OF NONCOMPACT EXTENSIONS OF HYPERBOLIC SYSTEMS

I. Melbourne
Department of Mathematics and Statistics
University of Surrey
Guildford, Surrey GU2 7XH, UK

V. Niţică
Department of Mathematics
West Chester University
West Chester, PA 19383, USA

A. Török
University of Houston
Department of Mathematics, 651 PGH
Houston, TX 77204-3008, USA

VN, AT: Institute of Mathematics
of the Romanian Academy, P.O. Box 1–764
RO-70700 Bucharest, Romania

Abstract. Let $f : X \to X$ be the restriction to a hyperbolic basic set of a smooth diffeomorphism. If G is the special Euclidean group $SE(2)$ we show that in the set of C^2 G-extensions of f there exists an open and dense subset of stably transitive transformations. If $G = K \times \mathbb{R}^n$, where K is a compact connected Lie group, we show that an open and dense set of C^2 G-extensions satisfying a certain separation condition are transitive. The separation condition is necessary.

1. **Introduction.** This paper is part of a program attempting to classify the obstructions to (stable) topological transitivity in various classes of skew-product transformations with non-compact fiber, which is part of the current surge of activity in the study of partially hyperbolic systems. Recall that if X is a topological space, and $f : X \to X$ a continuous map, then f is said to be *transitive* if it has a dense orbit.

Let M be a smooth manifold endowed with a Riemannian metric. Let $f : M \to M$ be a smooth diffeomorphism and $X \subset M$ a compact and f-invariant subset of M.

We say that $f : X \to X$ is *hyperbolic* if there exists a continuous Tf-invariant splitting $E^s \oplus E^u$ of the tangent bundle $T_X M$ and constants $C > 0, 0 < \lambda < 1$, such that for all $n \geq 0$ and $x \in X$ we have:

$$\|(Df^n)_x v\| \leq C\lambda^n \|v\|, \; v \in E^s$$
$$\|(Df^{-n})_x v\| \leq C\lambda^n \|v\|, \; v \in E^u.$$

2000 *Mathematics Subject Classification.* 37D20, 37D30.

Key words and phrases. Noncompact group extension, stable transitivity, skew-product, hyperbolic basic set.

The research of VN was supported in part by NSF Grant DMS 99-71826. The research of AT was supported in part by NSF Grant DMS-0244529. The research of IM, VN, AT was supported in part by EPSRC Grant GR/S11862/01.

We say that X is *maximal and isolated* if there exists an open neighborhood U of X such that every compact f-invariant set of U is contained in X.

The set X is a *basic set* for $f : M \to M$ if:

1. f is hyperbolic on X;
2. X is maximal and isolated;
3. $f : X \to X$ is transitive.

The basic set is nontrivial if it is not a periodic orbit. Throughout, we restrict attention to basic sets that are nontrivial.

Assume that X is a hyperbolic basic set for $f : X \to X$ and let G be a finite-dimensional connected Lie group. Let $\beta : X \to G$ be a continuous map, called a *cocycle*. Define a skew product, or G-extension,

$$f_\beta : X \times G \to X \times G, \quad f_\beta(x, h) = (f(x), \beta(x)h).$$

The G-extension f_β is called *stably transitive* if β lies in the interior (usually in the Hölder or C^r topology, $r \geq 1$) of the subset of extensions that are topologically transitive. The question of interest here is whether noncompact group extensions of a hyperbolic basic set are typically *stably* topologically transitive.

It was already observed in [12] that β taking values in a proper closed sub-semigroup S of G is an obstruction to (stable) transitivity. For example if $G = \mathbb{R}$ and the image of β is included in $\mathbb{R}^+$, or if $G = SL(2, \mathbb{R})$ and the image of β is included in the set of matrices with positive entries, then f_β cannot be transitive. In [11] it is conjectured that this is essentially the only obstruction.

Several classes of groups for which the conjecture has been verified are: G compact [1, 5]; $SE(n)$, $n \geq 4$ even [10]; Euclidean spaces $\mathbb{R}^n$ [13, 5]. In [11] we prove the existence of a transitive extension for the fiber being any connected Lie group. Moreover [11] contains examples of stably transitive extensions with fiber the symplectic groups $Sp(n, \mathbb{R})$, as well as other noncompact groups.

For direct products $G = K \times \mathbb{R}^n$ with K compact, and for the group $G = SE(2)$, the conjecture in [11] was verified when the base is a hyperbolic attractor. In this paper, we verify the conjecture for these groups when X is a general basic set.

Let $G = K \times \mathbb{R}^n$, where K is a compact connected Lie group. A (closed) *half-space* in $\mathbb{R}^n$ is a closed region bounded by a hyperplane passing through the origin. It is clear that f_β cannot be transitive if the $\mathbb{R}^n$-component of β is cohomologous to a cocycle with values in a half-space in $\mathbb{R}^n$. Hence, we define $\mathcal{S}$ to be the open set of C^r cocycles β for which the $\mathbb{R}^n$-component is *not* cohomologous to a cocycle with values in a half-space.

Theorem 1.1. *Let X be a hyperbolic basic set for $f : X \to X$ and $r \geq 2$. Suppose that $G = K \times \mathbb{R}^n$. Then there is a C^2-open and C^r-dense set of cocycles $\beta \in \mathcal{S}$ for which $f_\beta : X \times G \to X \times G$ is transitive.*

Recall that $SE(n) = SO(n) \ltimes \mathbb{R}^n$ is the group generated by rotations and translations in $\mathbb{R}^n$. The multiplication in $SE(n)$ is given by $(\theta_1, v_1)(\theta_2, v_2) = (\theta_1\theta_2, \theta_1 v_2 + v_1)$. When the fiber is $SE(n)$, no separation condition is necessary for transitivity.

Theorem 1.2. *Let X be a hyperbolic basic set for $f : X \to X$ and $r \geq 2$. Suppose that $G = SE(2)$. Then there is a C^2-open and C^r-dense set of C^r cocycles $\beta : X \to G$ for which $f_\beta : X \times G \to X \times G$ is transitive.*

Remark. As in [5], we also obtain C^r-open and C^r-dense sets of transitive cocycles for all $r > 0$ (with C^1 interpreted as Lipschitz). If X is an attractor, we proved Hölder open and C^r-dense for all $r > 0$ in [11].

The proof relies on a blend of techniques developed to study extensions with non-compact fibers. Some of these techniques are reviewed in Section 2. In Section 3, we prove Theorem 1.1 in the special case $K = C \times \mathbb{T}^d$ where C is a compact semisimple group and $\mathbb{T}^d$ is a torus. In Section 4, we complete the proof of Theorem 1.1. Theorem 1.2 is proved in Section 5.

2. **Preliminaries.** We review several results needed for the proof of the main results. In particular we review a couple of transitivity criteria that are useful for non-compact extensions.

Assume that X is a hyperbolic basic set for $f : X \to X$, G a Lie group, and $\beta : X \to G$ a Hölder cocycle. Denote $\beta_n(x) = \beta(f^{n-1}x)\ldots\beta(fx)\beta(x)$.

As in [13], for abelian G, we associate to each periodic orbit $f^k x = x$ the weight $\beta_k(x)$ and denote by $\mathcal{H}_\beta = \{\beta_k(x) : f^k x = x\}$ the collection of all weights assigned to the set of periodic orbits of f.

The following proposition is proved in Niţică & Pollicott [13, §6].

Proposition 2.1. *If G is abelian and $\mathcal{H}_\beta$ generates a dense semigroup in G, then f_β is transitive.*

Sketch of the proof. We need to show that for any nonempty open sets $U, V \subset X \times G$ there is a positive integer N such that $f_\beta^N(U) \cap V \neq \emptyset$. Let $(y, g_1) \in U$ and $(z, g_2) \in V$. Let $h = g_2 - g_1$. Let $\varepsilon > 0$ be fixed, and such that $B((y, g_1), \varepsilon) \subset U$ and $B((z, g_2), \varepsilon) \subset V$. Since $\mathcal{H}_\beta$ generates a dense semigroup, we can choose a finite set of periodic orbits $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_r$ such that the periodic weighting over them generates an ε-dense semigroup in G. Then one can use these periodic orbits and shadowing to build a trajectory for f_β that starts in $B((y, g_1), \varepsilon)$ and ends in $B((z, g_2), \varepsilon)$. $\square$

The following results that will be used later on are proved in Field *et al.* [5, Theorems 1.3, 1.7].

Theorem 2.2. *Let G be a compact connected Lie group and X a hyperbolic basic set for $f : X \to X$. For $r > 0$, there exists a C^r open and dense subset $\mathcal{W}_r$ of $C^r(M, G)$ such that for all $\beta \in \mathcal{W}_r$, the extension f_β is transitive.*

Theorem 2.3. *Let X be a hyperbolic basic set for $f : X \to X$, and $\mathcal{S}$ the subset of $\beta \in C^r(M, \mathbb{R}^n)$ satisfying the separation condition that $\mathcal{H}_\beta$ is not contained on one side of an $\mathbb{R}^n$ hyperplane passing through 0. For $r > 0$, there exists a C^r open and dense subset $\mathcal{W}_r$ of $\mathcal{S}$ such that for all $\beta \in \mathcal{W}_r$, the extension f_β is transitive.*

Next, we recall a technique due to Melbourne *et al.* [11]. Suppose for the moment that G is an arbitrary connected Lie group with Lie algebra LG. Choose a norm $\|\ \|$ on LG. There is a metric d on G with the following properties (Pollicott & Walkden [15, p. 2886]):

1. $d(\gamma_1\delta, \gamma_2\delta) = d(\gamma_1, \gamma_2)$;
2. $d(\delta\gamma_1, \delta\gamma_2) \leq \|\mathrm{Ad}(\delta)\| d(\gamma_1, \gamma_2)$;

for any $\gamma_1, \gamma_2, \delta \in G$. A basic estimate in terms of this metric is obtained in [11, Lemma 2.2]. For convenience, we restate this estimate in the simpler situation of this paper for groups of the form $G = K \ltimes \mathbb{R}^n$ where K is a compact connected Lie group. (Certain bunching and strong bunching conditions in [11] are automatic for such groups.)

Lemma 2.4. *Let $G = K \ltimes \mathbb{R}^n$ where K is a compact connected Lie group, and β an α-Hölder cocycle. There is a constant $C = C(f, \beta) > 0$ with the following property.*

Given $\varepsilon > 0$ sufficiently small and any $n \geq 1$, assume that there are two trajectories $x_k = f^k x_0$, $y_k = f^k y_0$ such that $d(x_k, y_k) < \varepsilon$ for $0 \leq k \leq n-1$. Then

$$d(\beta_n(x_0), \beta_n(y_0)) \leq C(\|\mathrm{Ad}(\beta_n(x_0))\| + 1)\varepsilon^\alpha.$$

As in [11], we associate to each $x \in X$ the set

$$\mathcal{L}_\beta(x) = \{g \in G | \text{ there are } x_k \in X, n_k > 0 \text{ such that } x_k \to x, f_\beta^{n_k}(x_k, e) \to (x, g)\}.$$

The set $\mathcal{L}_\beta(x)$ is a closed semigroup in G [11, Lemma 3.1].

The following proposition follows from [11, Theorem 3.3].

Proposition 2.5. *Let $G = K \ltimes \mathbb{R}^n$ where K is a compact connected Lie group. If there is a point $x \in X$ such that $\mathcal{L}_\beta(x) = G$, then the extension f_β is transitive.*

Sketch of proof. We need to show that for any nonempty open sets $U, V \subset X \times G$ there is a positive integer N such that $f_\beta^N(U) \cap V \neq \emptyset$. Let $(y, g_1) \in U$ and $(z, g_2) \in V$. Let $\varepsilon > 0$ be fixed, and such that $B((y, g_1), \varepsilon) \subset U$ and $B((z, g_2), \varepsilon) \subset V$. Let ω_1 be an orbit of f from $B(y, \varepsilon)$ to $B(x_0, \varepsilon)$, and ω_2 an orbit of f from $B(x_0, \varepsilon)$ to $B(z, \varepsilon)$. Using symbolic formalism, the orbits ω_1, ω_2 can be chosen to have reasonable length.

Since $\mathcal{L}_\beta(x_0) = G$, there exists an orbit ω of f starting and ending in $B(x_0, \varepsilon)$ such that $d(\beta(\omega), \beta(\omega_2)^{-1} h \beta(\omega_1)^{-1}) < \varepsilon$. Altogether, $\omega_1 \omega \omega_2$ gives a pseudo-orbit for f_β starting in U and ending in V. By standard shadowing techniques, one can find now an orbit $\widetilde{\omega}_1 \widetilde{\omega} \widetilde{\omega}_2$ of f which $K\varepsilon$-shadows the pseudo-orbit $\omega_1 \omega \omega_2$. The orbit $\widetilde{\omega}_1 \widetilde{\omega} \widetilde{\omega}_2$ has a lift to an orbit of f_β that starts in $B((y, g_1), \varepsilon)$ and ends in $B((z, g_2), \varepsilon)$. □

A proof of the following lemma follows from Appendix A of Niţică & Török [14].

Lemma 2.6. *Let $f : X \to X$ be a hyperbolic map, $G = K \ltimes \mathbb{R}^n$ and $\beta : X \to G$ a Hölder cocycle. Then f_β admits stable and unstable foliations on $X \times G$, which are Hölder and invariant under right-multiplication by elements of G. The stable leaf of f_β through (x, e) is the graph of the function*

$$\gamma_x^s : W^s(x) \to G, \qquad \gamma_x^s(t) = \lim_{n \to \infty} \beta_n(t)^{-1} \beta_n(x).$$

This function is α-Hölder, and varies continuously with the cocycle β in the following sense: if $\beta_k \to \beta$ in the C^0-topology and β_k stay C^α-bounded, then, on $W^s_{\mathrm{loc}}(x)$, $\gamma_{k,x}^s \to \gamma_x^s$ in the C^0-topology.

One way to generate elements in the set $\mathcal{L}_\beta(x)$ is given by [11, Lemma 4.1], which we describe below.

Lemma 2.7. *Let $x \in X$ be a fixed point for the transformation f and y a homoclinic point to x. If there is a subsequence $n_k \to \infty$ such that $\beta_{n_k}(x) \to e$, then $\omega_x(y) \in \mathcal{L}_\beta(x)$, where $\omega_x(y)$ is the holonomy of the homoclinic loop determined by y.*

Sketch of the proof. Let us describe the meaning of $\omega_x(y)$. Consider the homoclinic path determined by the orbit of $y \in W^s(x) \cap W^u(x)$ (covered along $W^u(x)$ from

x to y and then along $W^s(x)$ from y to x). Then, the lift to the unstable/stable foliations of f_β, with initial point (x, e), of this homoclinic path ends at $(x, \omega_x(y))$.

Applying Lemma 2.6 to f and f^{-1}, we obtain that the stable/unstable manifold are the graphs of the functions

$$\gamma_x^s : W^s(x) \to G, \qquad \gamma_x^s(t) = \lim_{n\to\infty} \beta_n(t)^{-1}\beta_n(x),$$
$$\gamma_x^u : W^u(x) \to G, \qquad \gamma_x^u(t) = \lim_{n\to-\infty} \beta_{-n}(t)^{-1}\beta_{-n}(x),$$

and the continuous dependence holds. Therefore, the holonomy around the homoclinic loop determined by $y \in W^s(x) \cap W^u(x)$ is

$$\begin{aligned}\omega_x(y) &= \lim_{n\to\infty} \left(\beta_n(y)^{-1}\beta_n(x)\right)^{-1} \beta_{-n}(y)^{-1}\beta_{-n}(x)\\ &= \lim_{n\to\infty} \beta_n(x)^{-1}\beta_{2n}(f^{-n}y)\beta_{-n}(x).\end{aligned}$$

Hence, if $\beta_{n_k}(x) \to e$, then $\omega_x(y) \in \mathcal{L}_\beta(x)$ because

$$\lim_{k\to\infty} f_\beta^{2n_k}(f^{-n_k}y, e) = \lim_{k\to\infty} (f^{n_k}y, \beta_{2n_k}(f^{-n_k}y)) = (x, \omega_x(y)).$$

□

Note that these holonomy values can be easily modified in a continuous way by changing β in an open set which contains only finitely many iterates of y.

We review next techniques of Brin [1, 2]. The group G is a semidirect product $K \ltimes \mathbb{R}^n$ where K is a compact connected Lie group (although, we will use these results for $G = K \times \mathbb{R}^n$ only).

The following definitions go back to Brin [2]. Denote by $\mathcal{W}^s(\xi)$, respectively $\mathcal{W}^u(\xi)$, the leaves through $\xi \in X \times G$ for the stable and unstable foliations in Lemma 2.6.

Definition 2.8. Let $f : X \to X$ be a hyperbolic map, and $\beta : X \to G$ a Hölder cocycle. A (u,s)-*path* between two points $\xi, \eta \in X \times G$ is a sequence of points $\xi_0 = \xi, \xi_1, \ldots, \xi_m = \eta \in X \times G$ such that $\xi_i \in \mathcal{W}^s(\xi_{i-1})$ or $\xi_i \in \mathcal{W}^u(\xi_{i-1})$. We will say that the pair ξ, η is (u,s)-*accessible*, and refer to ξ_i, $0 \le i \le m$, as *junction points.*

The pair of foliations $\mathcal{W}^s$ and $\mathcal{W}^u$ is called *accessible* if each pair of points of $X \times G$ is accessible.

A (u,s)-*rectangle at* x is a (u,s)-path of four segments that starts and ends in the fiber $\{x\}\times G$ (i.e., it is the lift of a stable-unstable closed rectangle in X starting at x).

Taking into account Lemma 2.6, the following result is a more precise statement of what is proven in [2, Theorem 1]. The claim $m > 0$ is implicit there (note that there is typo in [2, proof of Lemma 3]: $y_1 \in M_1$ should be $z_1 \in M_1$).

Lemma 2.9. *Let $G = K \ltimes \mathbb{R}^n$ where K is a compact connected Lie group, and let $\beta : X \to G$ be a Hölder cocycle over the hyperbolic map $f : X \to X$. Assume that the pair $\xi_1, \xi_2 \in X \times G$ is connected by a (u,s)-path whose junction points are nonwandering.*

Then, given any open neighborhoods $U_k \subset X \times G$ of ξ_k, $k = 1, 2$, there exists $m > 0$ such that $f_\beta^m(U_1) \cap U_2 \ne \emptyset$.

Corollary 2.10. *Let $f : X \to X$ and $\beta : X \to G$ be as in Lemma 2.9. Assume that each point of $X \times G$ is nonwandering. If the pair $(x, hg), (x, g) \in X \times G$ is (u,s)-accessible, then $h \in \mathcal{L}_\beta(x)$.*

Remark. This shows that for such skew products, the holonomy elements generated by (u, s)-rectangles at x (see the group H_C of Brin [2]) are contained in $\mathcal{L}_\beta(x)$.

The following lemma is a stronger version of [11, Theorem 5.10]. Its proof is as in [11], the only change being to replace [13, Lemma 5] by Lemma 2.12 below.

Lemma 2.11. *Let K be a compact connected Lie group and $S \subset K \times \mathbb{R}^n$. Assume that the projection of S onto $\mathbb{R}^n$ does not lie in a half-space. Then the closure of the semigroup generated by S is a group.*

Lemma 2.12. *Assume that the family $\mathcal{L} \subset \mathbb{R}^n$ does not lie in a half-space. Then the closed semigroup generated by $\mathcal{L}$ is a group.*

Proof. Denote by S the semigroup generated (without closure) by $\mathcal{L}$.

We claim that for each $v_0 \in \mathcal{L}$ there is a sequence of non-negative integers $m_k \to \infty$ and vectors $w_k \in S$ such that $m_k v_0 + w_k \to 0$. Hence $(m_k - 1)v_0 + w_k \to -v_0$ and so $-\mathcal{L}$ lies in the closure of S, yielding the conclusion.

To prove the claim, notice that because $\mathcal{L}$ is not contained in a half-space, there are nonzero vectors $v_1, \dots, v_\ell$ in $\mathcal{L}$ such that the origin is in the interior of the convex hull of $v_1, \dots, v_\ell$. Indeed, assume by contradiction that this is not the case. Then, for each finite $F \subset \mathcal{L} \setminus \{0\}$, the set H_F of halfspaces that contain F is nonempty. Clearly the family $\{H_F\}_F$ has the finite intersection property. Since the set of halfspaces is compact, there is a halfspace in the intersection of all the H_F's. This means that $\mathcal{L}$ is contained in a halfspace, a contradiction.

Pick $v_0 \in \mathcal{L}$. For $\alpha > 0$ small enough $-\alpha v_0$ is close to the origin, hence it is a convex combination of $v_1, \dots, v_\ell$. This gives a relation $\sum_{i=0}^{\ell} \alpha_i v_i = 0$ with $\alpha_0 > 0$, and $\alpha_i \geq 0$. Thus, $\sum_{i=0}^{\ell} (t\alpha_i) v_i = 0$ for each $t > 0$. Denote by $\widehat{t}$ the image of the vector $(t\alpha_0, t\alpha_1, \dots, t\alpha_\ell)$ in $\mathbb{T}^{\ell+1} = \mathbb{R}^{\ell+1}/\mathbb{Z}^{\ell+1}$. Because $\mathbb{T}^{\ell+1}$ is a compact group, there is a subsequence $t_k \to \infty$ for which $\widehat{t_k}$ converges to zero. Hence, denoting by $t_i^{(k)}$ the nearest integer to $t_k \alpha_i$, it follows that $m_k := t_0^{(k)}$ and $w_k := \sum_{i=1}^{\ell} t_i^{(k)} v_i$ have the desired property. □

3. **Groups $G = C \times A$ where C is compact semisimple and A is abelian.** In this section, we prove Theorem 1.1 for connected Lie groups of the form $G = C \times A$ where C is compact semisimple and A is abelian. Let $p_C : G \to C$ and $p_A : G \to A$ denote the canonical projections.

Writing $A = \mathbb{T}^d \times \mathbb{R}^n$, we denote by $\mathcal{S}$ the open set of C^r cocycles β $(r \geq 2)$ for which the $\mathbb{R}^n$-component is *not* cohomologous to a cocycle with values in a half-space.

Lemma 3.1. *Let $L \subset G$ be a closed semi-subgroup. Suppose that $p_A(L) = A$ and $\overline{p_C(L)} = C$. Then $L = G$.*

Proof. We can rewrite G in the form $G = K \times \mathbb{R}^n$ where $K = C \times \mathbb{T}^d$ is compact. Note that the set L satisfies the half-space hypothesis in Lemma 2.11 since $p_A(L) = A$. So it follows from Lemma 2.11 that L is a group.

Since $p_A(L) = A$, it suffices to show that $C \times \{e_A\} \subset L$. Now A is abelian and so $[L, L] = [p_C L, p_C L] \times \{e_A\}$. Therefore $\overline{[p_C L, p_C L]} \times \{e_A\} \subset L$. Since $p_C L$ is dense in C, it is immediate that $[p_C L, p_C L]$ is dense in $[C, C]$. But C is semisimple so $[C, C] = C$. Hence we have shown that $C \times \{e_A\} \subset L$ as required. □

Proposition 3.2. *Theorem 1.1 holds for connected Lie groups $G = C \times A$ where C is compact semisimple and A is abelian.*

Proof. Fix $x_0 \in X$ and consider the closed semigroup $\mathcal{L}_\beta(x_0) \subset G$. We construct a C^2 open and C^r dense set of C^r cocycles $\beta \in \mathcal{S}$ $(r \geq 2)$ such that $\mathcal{L}_\beta(x_0)$ satisfies the hypotheses of Lemma 3.1. Then $\mathcal{L}_\beta(x_0) = G$ and the result follows from Proposition 2.5.

By Theorem 2.3, in $\mathcal{S}$ there is an open and dense set $\mathcal{S}_1$ of cocycles $\beta : X \to G$ for which $f_{p_A\beta} : X \times A \to X \times A$ is transitive. Since C is compact, it is immediate that $p_A(\mathcal{L}_\beta(x_0)) = A$. It follows now from Lemma 2.11 that $\mathcal{L}_\beta(x_0)$ is a group and hence contains the identity element. It follows that (x_0, g) is nonwandering for all $g \in G$. Since x_0 is arbitrary, all points in $X \times G$ are nonwandering.

It follows from results on generators for compact semisimple Lie groups [9, 4] that there is an open and dense set of pairs of elements $(c_1, c_2) \in C \times C$ that generate C as a closed (semi)group. From [6, Lemma 2.9.1] we obtain an open and dense subset $\mathcal{S}_2$ of $\beta \in \mathcal{S}_1$ for which the (u, s)-rectangles of $p_C\beta$ at x_0 determine a pair of generators of C. Therefore, by Corollary 2.10, $p_C\mathcal{L}_\beta(x_0)$ contains a generator set of C for all extensions $\beta \in \mathcal{S}_2$. It follows that $\overline{p_C(\mathcal{L}_\beta(x_0))} = C$ for all these cocycles. This completes the proof. $\square$

4. **Groups $G = K \times \mathbb{R}^n$ where K is compact.** In this section, we complete the proof of Theorem 1.1.

Lemma 4.1. *Let G be a group of the form $G = K \times \mathbb{R}^n$ where K is a compact connected Lie group. Let X be a hyperbolic basic set for $f : X \to X$ and $\beta : X \to G$ a Hölder cocycle. Let $Y \subset X$ be a closed f-invariant subset. If the restriction $(f_\beta)|_{Y \times G} : Y \times G \to Y \times G$ is transitive, then $f_\beta : X \times G \to X \times G$ is transitive.*

Proof. Let $y_0 \in Y$. We can define a closed semigroup $\mathcal{L}_\beta(y_0) \subset G$ as in Section 2, but we can do it using all trajectories in X or just those in Y. Denote the two closed semigroups $\mathcal{L}^X_\beta(y_0)$ and $\mathcal{L}^Y_\beta(y_0)$ respectively. Clearly, $\mathcal{L}^Y_\beta(y_0) \subset \mathcal{L}^X_\beta(y_0)$. Now transitivity of $Y \times G$ implies that $\mathcal{L}^Y_\beta(y_0) = G$. Hence $\mathcal{L}^X_\beta(y_0) = G$, implying transitivity of $X \times G$ by Proposition 2.5. $\square$

Remark. Lemma 4.1 is similar in nature to a result of Field *et al.* [5, Proposition 6.1].

Proof of Theorem 1.1. It is always possible to embed a horseshoe inside a hyperbolic basic set [8, Theorem 6.5.5, Exercise 6.5.1]). By Lemma 4.1, it suffices to prove the result for the horseshoe. Hence, we may suppose from the beginning that X is totally disconnected.

By the structure theorem for compact Lie groups ([3, Ch. V, Theorem 8.1]) there is a finite cover $p : C \times \mathbb{T}^d \to K$ where C is compact semisimple and $\mathbb{T}^d$ is a torus. This extends by the identity to a finite cover $p : \widehat{G} \to G$ where $\widehat{G} = C \times \mathbb{T}^d \times \mathbb{R}^n$ is a group of the form studied in Section 3.

Let $\widehat{\mathcal{S}}$ consist of C^r cocycles $\hat{\beta} : X \to \widehat{G}$ for which the $\mathbb{R}^n$-component of $\hat{\beta}$ is not cohomologous to a cocycle with values in a half-space. Recall that the cocycles we are considering are C^r functions on a neighborhood of X in the ambient manifold, but we are only interested in their restrictions to X. Since X is totally disconnected, each C^r cocycle $\beta : X \to G$ has a C^r lift $\beta : X \to \widehat{G}$. Clearly $p^{-1}(\mathcal{S}) = \widehat{\mathcal{S}}$.

By Proposition 3.2, it follows that there is an open and dense set of transitive cocycles $\hat{\beta} : X \to \widehat{G}$ in $\widehat{\mathcal{S}}$. Clearly, if $\hat{\beta}$ defines a transitive $\widehat{G}$-extension, then $\beta = p\hat{\beta}$ defines a transitive G-extension.

Given $\beta \in \mathcal{S}$, let $\hat{\beta} \in \widehat{\mathcal{S}}$ be a lift of it to $\widehat{G}$. After an arbitrarily small perturbation, we may suppose that $\hat{\beta}$ defines a stably transitive $\widehat{G}$-extension. Moreover, cocycles β' that are Hölder close to β can be lifted to cocycles $\hat{\beta}'$ that are Hölder close to $\hat{\beta}$. Since $\hat{\beta}$ is stably transitive, the cocycles $\hat{\beta}'$ are transitive and hence the cocycles β' are transitive. Hence β is stably transitive as required. $\square$

5. $SE(2)$**-extensions.** In this section, we prove Theorem 1.2.

Lemma 5.1. *There is an open and dense set of pairs of elements $(g_1, g_2) \in SE(2)^2$ such that the closed semigroup H generated by g_1 and g_2 is a group of the form $H = K \ltimes \mathbb{R}^2$ for a closed subgroup $K \subset SO(2)$.*

Proof. Let $g_1, g_2 \in SE(2)$ be two rotations with distinct centers of rotation, such that g_1 has order at least seven. The set of such pairs is open and dense in $SE(2)^2$.

Since g_1, g_2 are compact, the closed semigroup and group they generate coincide. Denote this by H. Up to a conjugation, we may assume that $g_1 = (\theta_1, 0)$ and $g_2 = (\theta_2, v_2)$, where $\theta_1, \theta_2 \in SO(2)$ and $v_2 \in \mathbb{R}^2 \setminus \{0\}$. Then

$$\begin{aligned} g_3 = g_1 g_2 g_1^{-1} g_2^{-1} &= (\theta_1, 0)(\theta_2, v_2)(\theta_1^{-1}, 0)(\theta_2^{-1}, -\theta_2^{-1} v_2) \\ &= (e, \theta_1 v_2 - v_2) \in H \cap (\mathbb{R}^2 \setminus \{0\}). \end{aligned}$$

Thus the group H contains g_1, a rotation of order at least seven, and g_3, a non-trivial translation. Denote by H_0 the closed group generated by g_1, g_3. Then $H_0 = K_1 \ltimes L$ where $K_1 \subset SO(2)$ is the closed group generated by g_1, $|K_1| \geq 7$, and $L \subset \mathbb{R}^2$ is a closed K_1-invariant subgroup. We claim that $L = \mathbb{R}^2$. This implies that H is a semidirect product $K \ltimes \mathbb{R}^2$.

To prove the claim, note that by the crystallographic restriction [7, pages 70-81], L is not a lattice. If L is not $\mathbb{R}^2$, then it is isomorphic to $\mathbb{R} \times \mathbb{Z}$, and therefore the $\mathbb{R}$-direction must be preserved by K_1. This is impossible. $\square$

Proof of Theorem 1.2. Let $p : SE(2) \to SO(2)$ be the canonical projection. We start with an arbitrary cocycle. Choose a fixed point $x_0 \in X$ for f, or for a higher iterate of f, and two points $y_1, y_2 \in X$ homoclinic to x_0 that have mutually disjoint orbits. Then, by making C^r-small perturbations over small neighborhoods of y_1 and y_2 we can arrange that the holonomies $\omega_{x_0}(y_1), \omega_{x_0}(y_2) \in \mathcal{L}_\beta(x_0)$ are in the set prescribed by Lemma 5.1. Since our perturbation can be arbitrarily small, the holonomies depend continuously on the perturbation, and the set in Lemma 5.1 is open, it follows that there is an open dense set $\mathcal{U}_1$ of cocycles $\beta : X \to SE(2)$ for which the pair of holonomies $\omega_{x_0}(y_1), \omega_{x_0}(y_2) \in \mathcal{L}_\beta(x_0)$ belong to the open, dense set of pairs defined in Lemma 5.1. Therefore, $\mathbb{R}^2 \subset \mathcal{L}_\beta(x_0)$ for all cocycles in $\mathcal{U}_1$.

It follows from Theorem 2.2 that, for a dense and open set $\mathcal{U}_2$ of β, the $SO(2)$ extension given by $p\beta$ is transitive.

We will show that $\mathcal{L}_\beta(x_0) = SE(2)$ for cocycles in $\mathcal{U}_1 \cap \mathcal{U}_2$. Since $\mathbb{R}^2 \subset \mathcal{L}_\beta(x_0)$, we only have to prove that $SO(2) \subset \mathcal{L}_\beta(x_0)$.

Let $\theta \in SO(2)$. By the transitivity of the $p\beta$ extension, there is a sequence of orbits $\mathcal{P}_n = \{x_n, \dots, f^{p_n}(x_n)\}$ for which $x_n \to x_0$, $f^{p_n}(x_n) \to x_0$, and $p\beta_{p_n}(x_n) \to \theta$. Write $\beta_{p_n}(x_n) = (\theta_n, v_n) \in SO(2) \ltimes \mathbb{R}^2$. Since $\mathbb{R}^2 \subset \mathcal{L}_\beta(x_0)$, one can find a sequence $\mathcal{R}_n = \{y_n, \dots, f^{r_n}(y_n)\}$ of orbits such that $y_n \to x_0$, $f^{r_n}(y_n) \to x_0$, and

$d(\beta_{r_n}(y_n),(e,-v_n)) \to 0$. Denote by $\{z_n,\ldots,f^{p_n+r_n}(z_n)\}$ the orbit that shadows the pseudo-orbit $\{\mathcal{R}_n,\mathcal{P}_n\}$. By Lemma 2.4 and the definition of the metric d,

$$d(\beta_{p_n+r_n}(z_n),\beta_{r_n}(y_n)\beta_{p_n}(x_n)) \le d(\beta_{r_n}(f^{p_n}z_n),\beta_{r_n}(y_n)) + d(\beta_{p_n}(z_n),\beta_{p_n}(x_n))$$

is small. Therefore, $\beta_{p_n+r_n}(z_n) \to (\theta,0)$, which shows that $(\theta,0) \in \mathcal{L}_\beta(x_0)$. $\square$

REFERENCES

[1] M. I. Brin. Topological transitivity of a class of dynamical systems and frame flow on manifolds of negative curvature. *Func. Anal. and Appl.* 9 (1975) 9–19.

[2] M. I. Brin. The topology of group extensions of C systems. *Mat. Zametki* 18 (1975) 453–465.

[3] T. Bröcker, T. tom Dieck. Representations of Compact Lie Groups. *Grad. Texts in Math.* 98, Springer, New York, 1985.

[4] M. J. Field. Generating sets for compact semisimple Lie groups. *Proc. Amer. Math. Soc.* 127 (1999) 3361–3365.

[5] M. J. Field, I. Melbourne, A. Török. Stable ergodicity for smooth compact Lie group extensions of hyperbolic basic sets. *Ergod. Th. & Dynam. Sys.* 25 (2005) 517–551.

[6] M. J. Field, V. Niţică. Stable topological transitivity of skew and principal extensions. *Nonlinearity* 14 (2001) 1055–1069.

[7] D. Hilbert, S. Cohen-Vossen. Geometry and the Imagination. *Chelsea Publishing Company*, 1952.

[8] A. Katok, B. Hasselblatt. Introduction to the Modern Theory of Dynamical Systems. *Encyclopedia of Mathematics and its Applications* 54. Cambridge University Press, 1995.

[9] M. Kuranishi. Two element generations on semi-simple Lie groups. *Kodai Math. Sem. Report* (1949) 9–10.

[10] I. Melbourne, M. Nicol. Stable transitivity of Euclidean group extensions. *Ergod. Th. & Dynam. Sys.* 23 (2003) 611–619.

[11] I. Melbourne, V. Niţică, A. Török. Stable transitivity of certain noncompact extensions of hyperbolic systems. *Annales Henri Poincare* 6 (2005) 725–746.

[12] V. Niţică. A note about topologically transitive cylindrical cascades. *Israel Journal of Math.* 126 (2001) 141–156.

[13] V. Niţică, M. Pollicott. Transitivity of Euclidean extensions of Anosov diffeomorphisms. *Ergod. Th. & Dynam. Sys.* 25 (2005) 257–269.

[14] V. Niţică, A. Török. An open and dense set of stably ergodic diffeomorphisms in a neighborhood of a non-ergodic one. *Topology* 40 (2001) 259–278.

[15] M. Pollicott, C. P. Walkden. Livšic theorems for connected Lie groups. *Trans. Amer. Math. Soc.* 353 (2001) 2879–2895.

Received November 2004; revised February 2005.

E-mail address: ism@math.uh.edu
E-mail address: vnitica@wcupa.edu
E-mail address: torok@math.uh.edu