Clustering of Volatility in Variable Diffusion Processes

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Abstract

Increments in financial markets have anomalous statistical properties including fat-tailed distributions and volatility clustering (i.e., the autocorrelation functions of return increments decay quickly but those of the squared increments decay slowly). One of the central questions in financial market analysis is whether the nature of the underlying stochastic process can be deduced from these statistical properties. We have shown previously that a class of *variable diffusion processes* have fat-tailed distributions. Here we show analytically that such models also exhibit volatility clustering. To our knowledge, this is the first case where clustering of volatility is proven analytically in a model.

Our results are compatible with the viewpoint that variable diffusion processes are possible models for financial markets.

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I. INTRODUCTION

Geometric Brownian motion is a canonical model for the price of a commodity P_t as a function of time t [1, 2], describing the time evolution as

$$P_t = P_0 e^{\mu t + \sigma B_t}$$

where μ and $\sigma > 0$ are constants, and B_t is a standard Brownian motion. In this model, the return $X_t := \ln P_t$ of the commodity experiences stochastic dynamics

$$dX_t = \mu dt + \sigma dB_t.$$

Under this scenario, the stochastic increments $\Delta X_\tau = X_{t+\tau} - X_t$ over a time interval $(t, t+\tau)$ are independent of t and are distributed normally as $\mathcal{N}(\mu\tau, \sigma^2\tau)$. It has been realized over the last decade however that increments of financial instruments lie either on exponential or on fat-tailed distributions, and furthermore that there exist long-range auto-correlations [3, 4]. In this paper we model commodity prices or currency exchange rates by a stochastic differential equation

$$dX_t = \sigma(X_t, t) dB_t$$

with a place and time dependent diffusion coefficient given by $\sigma(x,t) = \sqrt{1 + \varepsilon \frac{x^2}{t}}$. This model incorporates the assumption that large changes in the price increases the volatility of the underlying commodity. It is also mathematically tractable and conceptually simple. Increments in return of the resulting process have been shown to lie on fat-tailed distributions [5]. Below we compute moments and auto-correlation functions for the process. Part of the motivation for this work was to show that such variable diffusion processes exhibit "clustering of volatility", i.e., the observation that sample autocorrelation functions of returns in financial markets may decay quickly but their absolute or squared increments decay much more slowly [6–8] .

The model studied here is an example of a *variable diffusion process* [5], where the standard deviation has the form $\sigma(u)$, with $u = x/\sqrt{t}$. When the stochastic process is initiated by $x = 0$ at $t = 0$, the dynamics of the distribution $W(x,t)$ of the random variable $X(t)$ at time t is given by the Fokker-Planck equation

$$\frac{\partial}{\partial t} W(x,t) = \frac{1}{2} \frac{\partial^2}{\partial x^2} \left(\sigma\left(\frac{x}{\sqrt{t}}\right)^2 W(x,t) \right).$$

For such variable diffusion processes, it has been shown that $W(x,t)$ scales according to [5]

$$W(x,t) = \frac{1}{\sqrt{t}}\mathcal{F}(u),$$

where $\mathcal{F}(u)$ is referred to as the scaling function. Furthermore, when $\sigma(u)$ is symmetric ($\sigma(u) = \sigma(-u)$),

$$\mathcal{F}(u) = \frac{1}{\sigma(u)^2}\exp\left(-\int^u dy\, \frac{y}{\sigma(y)^2}\right),$$

and $W(\cdot,t)$ is also symmetric. The moments $\mathbb{M}_n(t)$ of $W(x,t)$ scale according to

$$\mathbb{M}_n(t) \sim t^{n/2} \text{ for } n \text{ even}, \quad \mathbb{M}_n(t) = 0 \text{ for } n \text{ odd}.$$

Note that the Fokker-Planck formalism describes the evolution of the distribution $W(x,t)$; however, it is not possible to use it to calculate the behavior of auto-correlations. The latter requires a Langevin formulation of the process which is outlined here.

For the model $\sigma(u)^2 = 1 + \varepsilon u^2$ used here, the scaling function has the form [5]

$$\mathcal{F}(u) = \frac{c_\varepsilon}{(1+\varepsilon u^2)^{1+(1/2\varepsilon)}}.$$

Note that the even moments of order larger or equal to $\frac{1}{\varepsilon} + 1$ diverge. When the even moments exist, they are given, inductively, by

$$\mathbb{M}_{2n}(t) = \frac{t}{\frac{1}{2n-1} - \varepsilon}\mathbb{M}_{2n-2}(t), \tag{1}$$

with $\mathbb{M}_0(t) = 1$. Calculations below require the 4-th order moments to exist; thus it is necessary that $0 \le \varepsilon < 1/3$.

Autocorrelation functions in financial markets exhibit anomalous features as well [6, 7]. In particular, consider the autocorrelation function of the increments in the signal X_t

$$\mathcal{A}_1(t_1, t_2; \tau) := \mathrm{Cor}\left((X_{t_1+\tau} - X_{t_1}), (X_{t_2+\tau} - X_{t_2})\right),$$

and that of the square of the increments

$$\mathcal{A}_2(t_1, t_2; \tau) := \mathrm{Cor}\left((X_{t_1+\tau} - X_{t_1})^2, (X_{t_2+\tau} - X_{t_2})^2\right),$$

where the correlation function is defined in Eqn.(2) below. Note that, if the stochastic process has stationary increments, then $\mathcal{A}_1(t_1, t_2; \tau)$ and $\mathcal{A}_2(t_1, t_2; \tau)$ are functions of $t_2 - t_1$ and τ.

Most analyses of financial markets have been conducted under this stationarity assumption, and hence that $\mathcal{A}_1(t_1, t_2; \tau) = \mathcal{A}_1(t_1 - t_2; \tau)$ and $\mathcal{A}_2(t_1, t_2; \tau) = \mathcal{A}_2(t_1 - t_2; \tau)$. For returns X_t of (for example) currency rates, $\mathcal{A}_1(t_1 - t_2; \tau)$ has been shown to effectively vanish for $|t_1 - t_2|$ larger than 10 minutes, indicating that the signal is uncorrelated beyond this period. It has also been shown that $\mathcal{A}_2(t_1 - t_2; \tau)$ decays slowly as a function of $|t_1 - t_2|$. This phenomenon is referred to as *volatility clustering*.

A. The Main Result

The main result of the paper is that our variable diffusion processes exhibit volatility clustering. More precisely, as shown below, although increments of the variable diffusion process we consider are not stationary, the autocorrelation function has the asymptotics $\mathcal{A}_2(t_1, t_1 + T; \tau) \sim CT^{\varepsilon - 1}$ as $T \to \infty$, with a factor C that depends on t_1 and τ (as well as the parameters of the SDE).

II. CORRELATION COMPUTATIONS

Recall that the correlation function between two random variables X and Y is given by

$$\mathrm{Cor}(X, Y) := \frac{\mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}}, \quad \mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2. \tag{2}$$

It can be expressed using the covariance

$$\mathrm{Cov}(X, Y) := \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y),$$

since $\mathrm{Var}(X) = \mathrm{Cov}(X, X)$. In view of these formulas, the autocorrelation functions (or correlation coefficients) $\mathcal{A}_1(t_1, t_2; \tau)$ and $\mathcal{A}_2(t_1, t_2; \tau)$ can be computed using covariances. As shown below, the latter can be calculated using expected values of products of two or four X's:

$$\mathbb{E}(X_t X_{t+\tau}) \text{ and } \mathbb{E}(X_t X_{t+\tau} X_{t+\omega} X_{t+\lambda}),$$

where $0 \le \tau \le \omega \le \lambda$. Note that since X_t is a martingale, some of these expected values coincide. For example, if $0 \le \tau \le \omega \le \lambda$ then

$$\mathbb{E}(X_t X_{t+\tau} X_{t+\omega} X_{t+\lambda}) = \mathbb{E}(X_t X_{t+\tau} X_{t+\omega}^2).$$

A. The stochastic process

In order to calculate the expectations, we consider the stochastic process X^a, $a > 0$, defined by

$$dX_t^a = \sigma_a(X_t^a, t)dB_t, \quad X_0^a = 0, \quad \sigma_a(x,t) = \sqrt{1 + \varepsilon \frac{x^2}{t+a}}$$

We consider the case $a > 0$ because, in contrast to the SDE with $\sigma_0(x,t) = \sigma(x,t) = \sqrt{1+\varepsilon\frac{x^2}{t}}$ which is undefined as $t \to 0$, standard Itô calculus techniques apply to the SDE with diffusion $\sigma_a(x,t)$, $a > 0$. Whether the SDE has a (unique) solution for $a = 0$ and if the procedures used here are valid for this case will be studied elsewhere.

We note however that one can formally do the computations that follow also for $a = 0$. For $0 \le \varepsilon < 1/3$, the limit as $a \to 0^+$ of the formulas that we obtain coincide with their versions for $a = 0$. This is in agreement with the numerical simulations for the case $a = 0$ (see Figure 1).

B. Notations

We introduce the following abbreviation:

$$\mathbb{E}_{\alpha\beta\gamma\ldots}(t,\tau,\omega,\ldots) := \mathbb{E}(X_t^\alpha X_{t+\tau}^\beta X_{t+\tau+\omega}^\gamma \cdots),$$

that is, the subscripts of $\mathbb{E}$ indicate the exponents, and the variables $\tau, \omega, \ldots$ indicate the time-shifts. For example,

$$\mathbb{E}_{112}(t,\tau,\omega) = \mathbb{E}(X_t X_{t+\tau} X_{t+\tau+\omega}^2).$$

The corresponding expected values of X^a are denoted by $\mathbb{E}^a$. However, since most calculations are performed for X^a and carrying the superscript a is cumbersome, we will simplify the notation by representing it by X. Thus, for example, we will use X_t for X_t^a. Further, $\mathbb{E}^a(X_t X_{t+\tau}^3)$ stands for $\mathbb{E}(X_t^a (X_{t+\tau}^a)^3)$, and $\mathbb{E}_{112}^a(t,\tau,\omega)$ stands for $\mathbb{E}(X_t^a X_{t+\tau}^a (X_{t+\tau+\omega}^a)^2)$.

Unless mentioned otherwise, $\varepsilon, \tau, \omega, \lambda, \cdots \ge 0$.

C. Computation of $\mathbb{E}(A(X_{t+\tau}^a)^k)$

Assume that the random variable A is in $\mathcal{F}_t^a$ (i.e., depends on $X_s = X_s^a$ and B_s, $s \le t$). We want to compute $\mathbb{E}(A(X_{t+\tau}^a)^k) = \mathbb{E}^a(AX_{t+\tau}^k)$ for $t, \tau \ge 0$.

The idea is to write, using Itô's formula, an SDE for the stochastic process $\tau \mapsto (X^a_{t+\tau})^k$. This leads, given the expression of σ_a, to an equation that can be solved recursively for the expectations $\mathbb{E}^a(AX^k_{t+\tau})$. We present next the details.

As before, simplify the notation using $\widetilde{X}_\tau := X^a_{t+\tau}$. Then $\widetilde{X}_\tau$ satisfies the SDE

$$d\widetilde{X}_\tau = \sigma_a(\widetilde{X}_\tau, t+\tau)dB_\tau, \qquad \widetilde{X}_0 = X^a_t.$$

Denote by d_τ the (stochastic) differential with respect to τ. From Itô's formula, for $k \geq 2$:

$$\begin{aligned}
d_\tau \widetilde{X}^k_\tau &= k\widetilde{X}^{k-1}_\tau \, d_\tau\widetilde{X}_\tau + \frac{1}{2}k(k-1)\widetilde{X}^{k-2}_\tau[d_\tau\widetilde{X}_\tau]^2 \\
&= k\left(1+\varepsilon\frac{\widetilde{X}^2_\tau}{t+a+\tau}\right)^{1/2}\widetilde{X}^{k-1}_\tau dB_\tau + \frac{1}{2}k(k-1)\left(1+\varepsilon\frac{\widetilde{X}^2_\tau}{t+a+\tau}\right)\widetilde{X}^{k-2}_\tau d\tau.
\end{aligned}$$

Write this in integral form:

$$\widetilde{X}^k_\tau = \widetilde{X}^k_0 + \int k\left(1+\varepsilon\frac{\widetilde{X}^2_\tau}{t+a+\tau}\right)^{1/2}\widetilde{X}^{k-1}_\tau dB_\tau + \frac{1}{2}k(k-1)\int\left(1+\varepsilon\frac{\widetilde{X}^2_\tau}{t+a+\tau}\right)\widetilde{X}^{k-2}_\tau d\tau.$$

Multiply by A, take expected values noting that the integral of the dB_τ-part has zero expected value, and we obtain

$$\mathbb{E}^a(AX^k_{t+\tau}) = \mathbb{E}^a(AX^k_t) + \frac{1}{2}k(k-1)\int_0^\tau \mathbb{E}^a\left(A\left(1+\varepsilon\frac{X^2_{t+s}}{t+a+s}\right)X^{k-2}_{t+s}\right)ds$$

so

$$\frac{d}{d\tau}\mathbb{E}^a(AX^k_{t+\tau}) = \frac{k(k-1)}{2}\left[\mathbb{E}^a(AX^{k-2}_{t+\tau}) + \frac{\varepsilon}{t+a+\tau}\,\mathbb{E}^a(AX^k_{t+\tau})\right], \quad t,\tau \geq 0, \quad k \geq 2. \tag{3}$$

Now solve (3) with respect to τ, having initial condition at $\tau = 0$ given by $\mathbb{E}^a(AX^k_t)$.

For $k = 2$ and $\varepsilon \neq 1$ one obtains

$$\mathbb{E}^a(AX^2_{t+\tau}) = \frac{\mathbb{E}^a(A)}{1-\varepsilon}(t+a+\tau) + \left[\mathbb{E}^a(AX^2_t) - \frac{\mathbb{E}^a(A)}{1-\varepsilon}(t+a)\right]\left(\frac{t+\tau+a}{t+a}\right)^\varepsilon. \tag{4}$$

For $k = 3$ and $\varepsilon \neq 1/3$, using the fact that $\mathbb{E}^a(AX_{t+\tau}) = \mathbb{E}^a(AX_t)$ by the martingale property, one obtains

$$\mathbb{E}^a(AX^3_{t+\tau}) = \frac{3\mathbb{E}^a(AX_t)}{1-3\varepsilon}(t+a+\tau) + \left[\mathbb{E}^a(AX^3_t) - \frac{3\mathbb{E}^a(AX_t)}{1-3\varepsilon}(t+a)\right]\left(\frac{t+a+\tau}{t+a}\right)^{3\varepsilon}. \tag{5}$$

This method can be used recursively to evaluate $\mathbb{E}^a(AX^k_{t+\tau})$ for $k \geq 4$.

D. The moments: $\mathbb{E}^a(X_t^k)$

We use the method of section II C with $A = 1$ and $t = 0$. We relabel τ to t. Recall that $\mathbb{E}^a_k(t) = \mathbb{E}((X^a_t)^k)$ by the notation introduced earlier. Then (3) becomes

$$\frac{d}{dt}\mathbb{E}^a_k(t) = \frac{k(k-1)}{2}\left[\mathbb{E}^a_{k-2}(t) + \frac{\varepsilon}{t+a}\,\mathbb{E}^a_k(t)\right], \quad t > 0 \tag{6}$$

with $\mathbb{E}^a_k(0) = 0$.

We do not need odd moments, but one can notice that the first moment, $\mathbb{E}^a(X_t)$, is equal to $\mathbb{E}^a(X_0) = 0$ because X_t is a martingale. From (6), it then follows that all odd moments are zero.

For $k = 2$ and $\varepsilon \neq 1$, one has from (4) that

$$\mathbb{E}^a_2(t) = \frac{t+a}{1-\varepsilon} - \frac{a^{1-\varepsilon}(t+a)^{\varepsilon}}{1-\varepsilon} \tag{7}$$

Solving (6) for $k = 4$ and $\varepsilon \notin \{1/5, 1/3, 1\}$ one obtains

$$\mathbb{E}^a_4(t) = \frac{3\,(a+t)^2}{(1-3\,\varepsilon)\,(1-\varepsilon)} + \frac{3\,a^{2-6\,\varepsilon}\,(a+t)^{6\,\varepsilon}}{(1-5\,\varepsilon)\,(1-3\,\varepsilon)} - \frac{6\,a^{1-\varepsilon}\,(a+t)^{1+\varepsilon}}{(1-5\,\varepsilon)\,(1-\varepsilon)}. \tag{8}$$

For $\varepsilon = 1/5$ the solution is

$$\mathbb{E}^a_4(t) = \frac{3\,(a+t)^2}{(1-3\,\varepsilon)\,(1-\varepsilon)} + (a+t)^{6\,\varepsilon}\left(\frac{-3\,a^{2-6\,\varepsilon}}{1-3\,\varepsilon} + 6\,a^{1-3\,\varepsilon}\,\ln(a)\right) - \frac{6\,a^{1-\varepsilon}\,\ln(a+t)}{1-\varepsilon}. \tag{9}$$

Note that $\lim_{a\to 0^+}\mathbb{E}^a_2(t) = \mathbb{M}_2(t)$ and $\lim_{a\to 0^+}\mathbb{E}^a_4(t) = \mathbb{M}_4(t)$, where $\mathbb{M}_{2n}(t)$ are calculated using the scaling function $\mathcal{F}(u)$, see Eqn. (1). In general, it can be shown, inductively, that $\lim_{a\to 0^+}\mathbb{E}^a_n(t) = \mathbb{M}_n(t)$.

E. Other expected values

All the expected values that we need to calculate the correlation coefficients can be obtained from equations (4), (5), (7), (8) and (9). Indeed, denoting by $EE_2(t,\tau,\mathbb{E}^a(A),\mathbb{E}^a(AX_t^2))$ the expression given by (4) for $\mathbb{E}^a(AX^2_{t+\tau})$ and by $EE_3(t,\tau,\mathbb{E}^a(AX_t),\mathbb{E}^a(AX_t^3))$ the expression given by (5) for $\mathbb{E}^a(AX^3_{t+\tau})$, one has:

$$\mathbb{E}^a_{13}(t,\tau) = EE_3(t,\tau,\mathbb{E}^a_2(t),\mathbb{E}^a_4(t)) \qquad \mathbb{E}^a_{112}(t,\tau,\omega) = EE_2(t+\tau,\omega,\mathbb{E}^a_2(t),\mathbb{E}^a_{13}(t,\tau))$$

$$\mathbb{E}^a_{22}(t,\tau) = EE_2(t,\tau,\mathbb{E}^a_2(t),\mathbb{E}^a_4(t)) \qquad \mathbb{E}^a_{121}(t,\tau,\omega) = \mathbb{E}^a_{13}(t,\tau)$$

$$\mathbb{E}^a_{211}(t,\tau,\omega) = \mathbb{E}^a_{22}(t,\tau) \qquad \mathbb{E}^a_{31}(t,\tau) = \mathbb{E}^a_4(t)$$

$$\mathbb{E}^a_{1111}(t,\tau,\omega,\lambda) = \mathbb{E}^a_{112}(t,\tau,\omega) \qquad \mathbb{E}^a_{11}(t,\tau) = \mathbb{E}^a_2(t)$$

F. Computation of the auto-correlation functions

Assume $t_1 \le t_2$. We have to consider two cases, depending on the ordering of the values $\{t_1, t_1+\tau, t_2\}$, namely $t_1 \le t_2 \le t_1+\tau$ when the intervals overlap and $t_1 \le t_1+\tau \le t_2$ when they are disjoint. We first outline the computation of the autocorrelation function $\mathcal{A}_1^a(t_1,t_2;\tau)$. If $t_1 \le t_1+\tau \le t_2$,

$$\begin{aligned}\mathbb{E}^a[(X_{t_1+\tau}-X_{t_1})(X_{t_2+\tau}-X_{t_2})] &= \mathbb{E}^a[X_{t_1+\tau}X_{t_2+\tau}]-\mathbb{E}^a[X_{t_1}X_{t_2+\tau}]-\mathbb{E}^a[X_{t_1+\tau}X_{t_2}]+\mathbb{E}^a[X_{t_1}X_{t_2}] \\ &= \mathbb{E}^a[X_{t_1+\tau}^2]-\mathbb{E}^a[X_{t_1}^2]-\mathbb{E}^a[X_{t_1+\tau}^2]+\mathbb{E}^a[X_{t_1}^2]=0. \end{aligned} \tag{10}$$

The second equality follows from the martingale property. Consequently, $\mathcal{A}_1^a(t_1,t_2;\tau)$ vanishes. A similar calculation for $t_1 \le t_2 \le t_1+\tau$ gives that

$$\mathcal{A}_1^a(t_1,t_2;\tau)=\frac{\mathbb{E}^a[X_{t_1+\tau}^2]-\mathbb{E}^a[X_{t_2}^2]}{\sqrt{\mathbb{E}^a[X_{t_1+\tau}^2]-\mathbb{E}^a[X_{t_1}^2]}\sqrt{\mathbb{E}^a[X_{t_2+\tau}^2]-\mathbb{E}^a[X_{t_2}^2]}}.$$

Thus, $\mathcal{A}_1^a(t_1,t_2;\tau)$ is 1 when $t_2=t_1$, decreases to 0 as t_2 increases to $t_1+\tau$, and remains there for $t_2 > t_1+\tau$.

$\mathcal{A}_2^a(t_1,t_2;\tau)$ can also be calculated considering the overlapping and non-overlapping cases. Introduce

$$\begin{aligned}\mathcal{C}ov_2^a(t_1,t_2;\tau) &:= \mathrm{Cov}[(X_{t_1+\tau}^a-X_{t_1}^a)^2,(X_{t_2+\tau}^a-X_{t_2}^a)^2] \\ &= \mathbb{E}[(X_{t+\tau}^a-X_t^a)^2(X_{t+2\tau+\omega}^a-X_{t+\tau+\omega}^a)^2]-\mathbb{E}[(X_{t+\tau}^a-X_t^a)^2]\mathbb{E}[(X_{t+2\tau+\omega}^a-X_{t+\tau+\omega}^a)^2].\end{aligned}$$

For disjoint time intervals, i.e. $t_1 < t_1+\tau < t_2 < t_2+\tau$, denote $t_1=t$ and $t_2=t+\tau+\omega$ with $\omega>0$. The times are $t_1=t<t+\tau<t_2=t+\tau+\omega<t+2\tau+\omega$ and therefore

$\mathcal{C}ov_2^a(t_1,t_2;\tau) = \mathbb{E}_{22}^a(t+\tau,\omega+\tau)+\mathbb{E}_{22}^a(t+\tau,\omega)-2\,\mathbb{E}_{211}^a(t+\tau,\omega,\tau)-2\,\mathbb{E}_{112}^a(t,\tau,\omega+\tau)-2\,\mathbb{E}_{112}^a(t,\tau,\omega)+4\,\mathbb{E}_{1111}^a(t,\tau,\omega,\tau)+\mathbb{E}_{22}^a(t,\omega+2\,\tau)+\mathbb{E}_{22}^a(t,\omega+\tau)-2\,\mathbb{E}_{211}^a(t,\omega+\tau,\tau)-(-2\,\mathbb{E}_{11}^a(t,\tau)+\mathbb{E}_2^a(t)+\mathbb{E}_2^a(t+\tau))\,(-2\,\mathbb{E}_{11}^a(\omega+t+\tau,\tau)+\mathbb{E}_2^a(\omega+t+\tau)+\mathbb{E}_2^a(\omega+t+2\,\tau))\,.$

A similar expression can be derived for the case of overlapping intervals, $t_1 \le t_2 \le t_1+\tau$. In particular,

$$\mathrm{Var}((X_{t+\tau}^a-X_t^a)^2)=\mathcal{C}ov_2^a(t,t;\tau).$$

The expression for $\mathcal{A}_2^a(t_1,t_2;\tau)$ is complicated, but one can compute its behavior. It begins at 1 when $t_2=t_1$, decreases rapidly to a positive value as t_2 increases to $t_1+\tau$, and then decreases very slowly as t_2 grows further. If $1/3>\varepsilon>0$, the asymptotic behavior is

$$\mathcal{A}_2^a(t,t+T;\tau)\sim CT^{\varepsilon-1} \quad \text{as} \quad T\to\infty.$$

Indeed, the numerator of $\mathcal{A}_2^a(t, t+T; \tau)$ behaves as $T^{\varepsilon-1}$ and its denominator converges to a constant:

$$\mathcal{C}ov_2^a(t, T; \tau) = C_{t,\tau;\varepsilon,a}\left[(a+T+\tau)^\varepsilon - (a+T)^\varepsilon\right] = C_{t,\tau;\varepsilon,a}\left(\varepsilon\tau T^{\varepsilon-1} + O(T^{\varepsilon-2})\right)$$

$$\lim_{t\to\infty} \mathrm{Var}((X_{t+\tau}^a - X_t^a)^2) = \frac{2\tau^2\,(1-3\varepsilon+3\varepsilon^2)}{(1-\varepsilon)^2(1-3\varepsilon)}$$

The value $C_{t,\tau;\varepsilon,a}$ is a complicated expression that can be computed explicitly. For $a=0$, $t>0$, it simplifies to

$$C_{t,\tau;\varepsilon,0} = \frac{2\,(t^2 - (t+\tau)^\varepsilon t^{2-\varepsilon} - \tau^2)}{(1-\varepsilon)^2(1-3\varepsilon)(t+\tau)^\varepsilon}$$

The dashed line of Figure 1 shows the behavior of $\mathcal{A}_2^a(t, t+T; \tau)$ as a function of T for $t = 10$, $\varepsilon = 0.1$, $\tau = 1$ and $a = 0.001$. The red dots are mean values from a series of 10^8 variable diffusion stochastic trajectories starting from the origin at $t = 0$ with $\sigma(x,t) = \sigma_0(x,t) = \sqrt{1+\varepsilon\frac{x^2}{t}}$. As $T \to \infty$, the autocorrelation function behaves like $\mathcal{A}_2^a(10, 10+T; 1) \sim T^{\varepsilon-1}$, in agreement with the derivation above.

Financial markets have been reported to exhibit volatility clustering [6]. We compute autocorrelation functions for the Euro-Dollar exchange rates during 1999-2005 recorded in 1-min intervals. As shown in Ref. [13], mean intraday increments in this market are time dependent, except in a time interval of approximately 500 minutes beginning at 20 hours GMT. We, therefore, limit our analysis to this time interval. The autocorrelation function for increments vanishes for $T > 10$ and, as shown in Figure 2, the autocorrelation function $\mathcal{A}_2(T_0, T; 10)$ exhibits a power-law decay, in qualitative agreement with the calculations presented here.

III. CONCLUSIONS

Analyses of many financial instruments over the past decade [3–5, 9–12] have repeatedly shown that increments of their return share two common anomalous characteristics: (1) The distribution of increments over an interval τ scales in τ, but is either exponential or has power-law tails. (2) The increments exhibit volatility clustering.

Using the Fokker-Planck formalism, it was previously shown that scaling distributions with fat-tails can arise from variable diffusion processes [5]. These stochastic differential equations have a diffusion coefficient that is a function of $u = x/\sqrt{t}$.

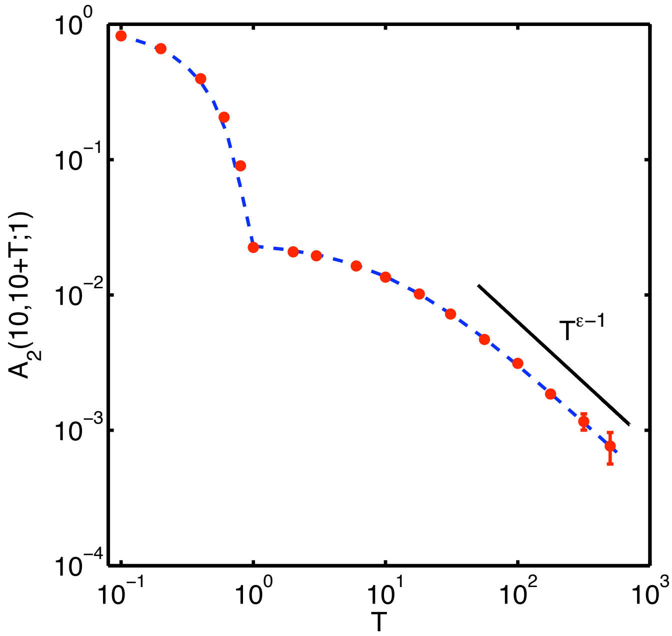

FIG. 1: The autocorrelation function of squared increments. The blue dashed line denotes the analytically derived autocorrelation function $\mathcal{A}_2^a(t, t+T; \tau)$ for $t = 10$, $\varepsilon = 0.1$, $\tau = 1$ and $a = 0.001$. For $T \to \infty$, $\mathcal{A}_2^a(t, t+T; \tau)$ decays like $T^{\varepsilon-1}$ shown by the black line. Red dots represent the corresponding autocorrelation function computed from a series of 10^8 stochastic trajectories starting from the origin at $t = 0$ with $\sigma_0(x,t) = \sqrt{1 + \varepsilon \frac{x^2}{t}}$. Error bars are negligible except at the last two points.

In this paper, we introduced a method to calculate expected values and correlation coefficients of one of these variable diffusion process with $\sigma^2 = 1 + \varepsilon x^2/t$. We showed that the stochastic process exhibits clustering of volatility. Our results, coupled with the fact that the corresponding scaling distribution has fat-tails, suggest that this variable diffusion process may describe the dynamics of financial markets.

IV. ACKNOWLEDGMENT

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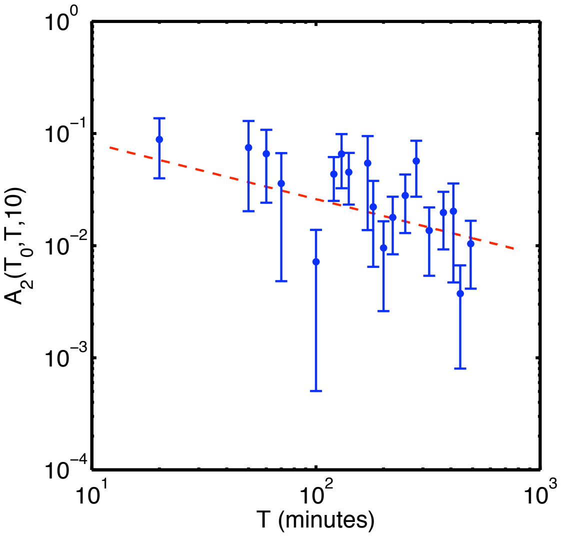

FIG. 2: The decay of $\mathcal{A}_2(T_0, T; \tau)$ for the Euro-Dollar exchange rates during 1999-2005. These calculations are conducted for a time interval of 500 minutes starting from 20 hours GMT, during which time the mean increments are stationary. We have chosen $T_0 = 10$ minutes and $\tau = 10$ minutes. The errors bars are the standard errors for the ≈ 1750 trading days in the data set.

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