AN OPEN DENSE SET OF STABLY ERGODIC DIFFEOMORPHISMS IN A NEIGHBORHOOD OF A NON-ERGODIC ONE

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Abstract. As a special case of our results we prove the following. Let $A \in \mathrm{Diff}^r(M)$ be an Anosov diffeomorphism. Then there is a C^r-neighborhood of $A \times \mathrm{Id}_{S^1}$ that contains an open dense set of partially hyperbolic diffeomorphisms that have the accessibility property. If, in addition, A preserves a smooth volume ν and λ is the Lebesgue measure on S^1, then in a neighborhood of $A \times \mathrm{Id}_{S^1}$ in $\mathrm{Diff}^2_{\nu\times\lambda}(M \times S^1)$ there is an open dense set of (stably) ergodic diffeomorphisms. Similar results are true for a neighborhood of the time-1 map of a topologically transitive (respectively volume preserving) Anosov flow. These partially answer a question posed by C. Pugh and M. Shub. We also describe an example of an accessible partially hyperbolic diffeomorphism that is not topologically transitive. This answers a question posed by M. Brin.

§1. Introduction

Let X be a compact smooth manifold without boundary endowed with a smooth volume μ. Denote by $\mathrm{Diff}^r(X)$ the set of C^r-diffeomorphisms of X, and by $\mathrm{Diff}^r_\mu(X)$ the set of volume preserving C^r-diffeomorphisms of X. A diffeomorphism $f \in \mathrm{Diff}^r_\mu(X)$ is called *ergodic* if for any measurable f-invariant set $A \subset X$ either $\mu(A) = 0$ or $\mu(X - A) = 0$. A diffeomorphism $f \in \mathrm{Diff}^r_\mu(X)$ is called *stably ergodic* if there is a neighborhood of f in $\mathrm{Diff}^r_\mu(X)$ consisting only of ergodic diffeomorphisms.

Remark. If no other mention is made, the notations C^r and Diff^r are used for the case $r \geq 1$.

Classical results of Kolmogorov, Arnold and Moser show that stable ergodicity is not always a generic property in $\mathrm{Diff}^r_\mu(X)$. Indeed, by the KAM theory, near a non-degenerate integrable Hamiltonian system there is a set of positive measure of invariant tori.

In contrast, Anosov showed in [A] that a volume preserving uniformly hyperbolic diffeomorphism is stably ergodic. Grayson, Pugh and Shub [GPS] found the first example of a non-hyperbolic stably ergodic diffeomorphism, the time-1 map of the geodesic flow of a surface S of constant negative curvature. Wilkinson [W] generalized this result to the

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case when S has variable negative curvature. Pugh and Shub [PS1] later proved it for higher dimensional manifolds of (almost) constant negative curvature. These results were improved in [PS2].

Results about genericity of ergodic skew-products over Anosov diffeomorphisms with a compact connected Lie group as fiber were found by Brin in [B]. These rely on earlier work of Brin and Pesin [BP] about ergodic partially hyperbolic diffeomorphisms. Recently Burns and Wilkinson [BW] proved results about the stable ergodicity of ergodic skew-products with the fiber a compact connected Lie group.

Recall that a partially hyperbolic diffeomorphism has stable and unstable foliations, W^s and W^u. In addition, an r-normally hyperbolic diffeomorphism has also an invariant center "leaf-immersion" with C^r leaves, tangent to the center distribution. (For the precise definitions see §2.) Given a partially hyperbolic diffeomorphism $f \in \mathrm{Diff}(X)$, we say that two points $x, y \in X$ are accessible (or, more precisely, (u, s)-accessible) if they can be joined by a piecewise differentiable piecewise nonsingular path consisting of segments tangent to either E^u or E^s. *(Essential) accessibility* of f means that (almost) each pair of points $x, y \in X$ is accessible. We say that a set $A \subset X$ is accessible if any two points $x, y \in A$ are accessible.

It was conjectured by C. Pugh and M. Shub that the set of stably ergodic diffeomorphisms is open and dense among the partially hyperbolic C^2 volume preserving diffeomorphisms of a compact manifold X. They conjectured also that stable accessibility is an open and dense property among C^2 partially hyperbolic diffeomorphisms, volume preserving or not. (Note that in both conjectures the openness follows from the definitions.)

Accessibility is relevant due to the following remarkable result of Pugh and Shub [PS2]:

Theorem 1.1. *(Pugh, Shub) If $f \in \mathrm{Diff}^2_\mu(X)$ is a center bunched and dynamically coherent partially hyperbolic diffeomorphism with the essential accessibility property then f is ergodic.*

This paper provides some evidence for a positive answer for the latter conjecture. Our main result shows that in an open class of partially hyperbolic diffeomorphisms, there is an open and dense set consisting of diffeomorphisms that have the accessibility property:

Theorem 1.2. *In both $\mathrm{Diff}^r(X)$ and $\mathrm{Diff}^r_\mu(X)$ the stably accessible partially hyperbolic diffeomorphisms form a C^1-open and C^r-dense set among those r-normally hyperbolic diffeomorphisms with 1-dimensional center distribution that have two compact periodic leaves whose Hausdorff distance is small enough. (By Theorem 6.1 of* [HPS]*, the latter condition is open in C^r.)*

More precisely, we need the following (C^r-open) conditions on the center leaves:

— there is a periodic compact 1-dimensional center leaf $\mathcal{C}_0$, and

— each point of $\mathcal{C}_0$ is close enough to a periodic compact center leaf that is disjoint from $\mathcal{C}_0$ (by compactness, finitely many such leaves fulfill this condition for all the points of $\mathcal{C}_0$).

The precise meaning of "nearby" and "close enough" is given by Definition 2.1.

Theorem 1.2 follows from Theorems 3.2 and 3.3. In the former we exhibit a C^1-open set of partially hyperbolic diffeomorphism that are accessible. In the latter we show that some partially hyperbolic diffeomorphisms can be made to belong to this set by an arbitrarily

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C^r-small perturbation. If a smooth volume is given, the perturbation can be made volume preserving.

In some cases the existence of periodic center leaves follows from the Poincaré Recurrence Theorem and a Shadowing Lemma in [HPS]:

Lemma 1.3. *Assume X is a compact manifold endowed with a probability measure μ that is positive on open sets. Let $f \in \mathrm{Diff}^1_\mu(X)$ be a partially hyperbolic diffeomorphism that is dynamically coherent and whose center lamination is plaque expansive. Then the periodic center leaves of f are dense in X.*

A proof of this fact will be sketched in Appendix 2.

Therefore, Theorems 1.1 and 1.2 imply that the set of stably ergodic partially hyperbolic diffeomorphisms is open and dense in certain open sets of $\mathrm{Diff}^2_\mu(X)$:

Theorem 1.4. *In $\mathrm{Diff}^2_\mu(X)$ the stably ergodic partially hyperbolic diffeomorphisms form an open and dense set among those diffeomorphisms that are 2-normally hyperbolic, center bunched, dynamically coherent, plaque expansive, with 1-dimensional center distribution, and have compact center leaves. (By Theorem 6.1 of [HPS] and Proposition 2.3 of [PS1], the latter set is open in C^2.)*

The main examples consist of time-1 maps of Anosov flows, and of $\mathrm{Diff}(S^1)$-valued skew-products over Anosov diffeomorphisms:

Theorem 1.5. *Let ϕ_1 be the time-1 map of a C^r Anosov flow $\{\phi_t\}$ of a compact manifold M. Then the following are true:*

(a) *Assume that the flow $\{\phi_t\}$ is topologically transitive. Then there is a C^r-neighborhood of ϕ_1 in $\mathrm{Diff}^r(M)$ that contains an open and dense set of diffeomorphisms with the accessibility property.*
(b) *Assume that there is a $\{\phi_t\}$-invariant volume μ on M. Then there is a C^2-neighborhood of ϕ_1 in $\mathrm{Diff}^2_\mu(M)$ that contains an open and dense set of stably ergodic diffeomorphisms.*

Indeed, for time-1 maps of Anosov flows that are topologically transitive or volume preserving, the compact center leaves (i.e., the closed trajectories of the flow) are dense. This follows from Anosov's Closing Lemma. Note that, in general, the time-1 map of an Anosov flow is not accessible. For example, in the case of a suspension flow over an Anosov diffeomorphism, the strong stable and strong unstable distributions are jointly integrable.

Theorem 1.6. *Let A be a C^r Anosov diffeomorphism of a compact manifold M and $\beta : M \to \mathrm{Diff}^r(S^1)$ a C^r map (i.e., $(x,y) \in M \times S^1 \mapsto \beta(x)(y) \in S^1$ is C^r). Assume that*

$$\|T^s A\| < \inf_{x\in M} m(T\beta(x)), \quad \textit{and} \quad \sup_{x\in M} \|T\beta(x)\| < m(T^u A). \tag{1.1}$$

Let $f : M \times S^1 \to M \times S^1$ be given by the skew-product $f(x,y) := (Ax, \beta(x)(y))$. Then the following is true:

(a) *There is a C^r-neighborhood of f in $\mathrm{Diff}^r(M \times S^1)$ that contains an open and dense set of diffeomorphisms with the accessibility property.*

(b) *Assume moreover that f is center bunched and there is a smooth f-invariant volume μ on $M \times S^1$. Then there is a C^2-neighborhood of f in $\mathrm{Diff}^2_\mu(M \times S^1)$ that contains an open and dense set of stably ergodic diffeomorphisms.*

(The assumption (1.1) implies that the skew-product f is a partially hyperbolic diffeomorphism. For the definition of the conorm $m(\cdot)$ see §2.)

Corollary 1.7. *Let ν be a smooth volume form on M and λ the Lebesgue measure on S^1. If $A \in \mathrm{Diff}^2_\nu(M)$ is hyperbolic then in a C^2-neighborhood of $A \times \mathrm{Id}_{S^1} \in \mathrm{Diff}^2_{\nu\times\lambda}(M \times S^1)$ there is an open dense set of ergodic diffeomorphisms.*

A related question is the topological transitivity of partially hyperbolic diffeomorphisms. Recall that a homeomorphism f of a topological space X is topologically transitive if for some point $x \in X$ the orbit $\{f^n(x) \mid n \in \mathbb{Z}\}$ is dense. If X is a compact metrizable perfect space then topological transitivity is equivalent to the existence of a point $y \in X$ whose *positive semi-orbit* $\{f^n(y) \mid n = 0, 1, 2 \dots\}$ is dense (see Exercise 1.4.2 in [KH]).

A diffeomorphism of a compact Riemannian manifold that has an ergodic invariant volume is topologically transitive. Therefore, by (b) of Theorem 1.6, there is an open and dense set of transitive diffeomorphisms in a C^2-neighborhood of $A \times \mathrm{Id}_{S^1}$ in $\mathrm{Diff}^2_\mu(M \times S^1)$.

Brin proved in [B] that a partially hyperbolic diffeomorphism that has the accessibility property and is recurrent is also topologically transitive. He asked if accessibility of a partially hyperbolic diffeomorphism is sufficient for topological transitivity. Our next corollary answers this question in the negative due to the fact that, if A is an Anosov diffeomorphism, the set of topologically transitive diffeomorphisms is not dense in any C^1-neighborhood of $A \times \mathrm{Id}_{S^1}$. We are grateful to K. Burns for pointing out this consequence of our results and to the referee for suggesting that we state it in this generality.

Corollary 1.8. *Let A be a C^1 Anosov diffeomorphism of a compact manifold M and R a rotation of S^1. Then there exists arbitrarily C^1-close to $A \times R$ a C^1-open set of partially hyperbolic diffeomorphisms that are accessible, but not topologically transitive.*

Proof. By (a) of Theorem 1.6 it is enough to find arbitrarily close to $A \times R$ an open set of non-transitive diffeomorphisms.

For simplicity, we consider first the case $R = \mathrm{Id}_{S^1}$. Pick $\phi \in \mathrm{Diff}^1(S^1)$ as close to Id_{S^1} as desired such that ϕ has a fixed attracting point. Hence, there are open sets $U, V \subset S^1$, $\emptyset \neq \overline{V} \subsetneq U$ such that $\phi(\overline{U}) \subset V$. Let $f := A \times \phi$. Then $f(M \times \overline{U}) \subset M \times V$, and any map that is C^0-close to f has the same property. But such a transformation cannot be topologically transitive because each positive semi-orbit has at most one element in the open set $(M \times U) \setminus (M \times \overline{V})$.

For the case of a arbitrary rotation, let R' be a rational circle rotation close to R. There exists a family of disjoint open intervals $\{I_i\}_i$ such that $\cup_i \overline{I_i} \neq S^1$ and $R'(I_i) = I_{i+1}$, where $i \in \mathbb{Z}/(n\mathbb{Z})$ for some n. One can choose now arbitrarily close to R' a circle diffeomorphism ϕ for which there exist n disjoint open intervals $\{J_i\}_i$ such that $\emptyset \neq \overline{J_i} \subsetneq I_i$ and $\phi(\overline{I_i}) \subset J_{i+1}$. Let $f = A \times \phi$. Then $f(M \times \overline{I_i}) \subset M \times J_{i+1}$, and any map that is C^0-close to f has the same property. As above, such a map cannot be topologically transitive. □

On the other hand, Bonatti and Díaz [BN] proved, among other striking results, that if M and N are compact boundaryless manifolds and A is a transitive C^∞ Anosov diffeomorphism of M, then there is a C^∞-arc $F_t \in \mathrm{Diff}^\infty(M \times N)$, $t \in [0, 1]$, such that

$F_0 = A \times \mathrm{Id}_N$ and there is a C^1-neighborhood of $\{F_t\}_{t\in(0,1]}$ consisting of non-hyperbolic transitive diffeomorphisms.

Let us also mention that according to [GPS], finding a C^1-neighborhood of the time-1 map of the geodesic flow of a surface of constant negative curvature consisting of topologically transitive diffeomorphisms is an open question.

As we pointed out earlier, one can study accessibility and ergodicity in thinner classes of transformations, e.g., skew-products over Anosov diffeomorphisms. The proof of Theorem 1.2 can be easily adapted to show that accessibility is generic in a class of Hölder extensions of Anosov diffeomorphisms (see Theorem 4.1). We also extend Brin genericity result about topological transitivity to the class of Hölder cocycles (Theorem 4.2). Note that the corresponding result for ergodicity is proved in [FP]. We discuss these results in Appendix 1. The existence of the stable and unstable foliations for such transformations does not follow in a straightforward way from the general theory of partially hyperbolic diffeomorphisms, so we prove their existence as well.

§2. Definitions

We recall first several standard facts about partially hyperbolic diffeomorphisms. Given a linear transformation L between two normed linear spaces, the *norm* and the *conorm* of L are defined by

$$\|L\| := \sup\{\|Lv\| \mid \|v\| = 1\} \qquad \text{and} \qquad m(L) := \inf\{\|Lv\| \mid \|v\| = 1\} = \|L^{-1}\|^{-1}.$$

Remark. In the sequel by a C^r*-lamination*, $r > 0$, we mean a C^0-foliation whose leaves are immersed C^r-submanifolds that vary continuously in the C^r-topology. A *foliation* stands for a C^0-foliation.

Let X be a compact, connected, boundaryless manifold. Denote by TX the tangent bundle of X. A C^1-diffeomorphism $f : X \to X$ is called *partially hyperbolic* if the derivative $Tf : TX \to TX$ leaves invariant a continuous splitting $TX = E^s \oplus E^c \oplus E^u$, $E^s \neq 0 \neq E^u$, such that, with respect to a fixed Riemannian metric on TX:

$$\begin{gathered} \|T^u f^{-1}\| < 1, \quad \|T^s f\| < 1, \\ \|T^s_p f\| < m(T^c_p f), \quad \|T^c_p f\| < m(T^u_p f) \quad \text{for all } p \in X. \end{gathered} \tag{2.1}$$

(This is a slightly weaker condition than the one used in [PS2].) E^s, E^c and E^u are called the *stable*, *center*, respectively *unstable* distributions. If the center distribution $E^c = 0$, then f is called an *Anosov* (or *hyperbolic*) diffeomorphism.

The distributions E^s and E^u are tangent to unique Hölder laminations W^s and W^u which have C^1 leaves. These are called the *stable* and *unstable* foliations. [To be precise, we could call these *laminations*, but that is not the standard terminology.] If $E^u \oplus E^c$, E^c, and $E^c \oplus E^u$ are also tangent to continuous foliations with C^1 leaves W^{cu}, W^c, respectively W^{cs}, and if W^c and W^u subfoliate W^{cu}, while W^c and W^s subfoliate W^{cs}, then f is said to be *dynamically coherent*.

Let $f : X \to X$ be a partially hyperbolic diffeomorphism. f is r*-normally hyperbolic* if the center distribution E^c is integrable to a C^r-boundaryless leaf immersion (see [HPS],

§6) and

$$m(T_p^u f) \geq \|T_p^c f\|^k, \qquad \|T_p^s f\| \leq m(T_p^c f)^k, \quad k = 0, \ldots, r.$$

Roughly speaking, the center distribution integrates to a "lamination" that can have self-intersections; its leaves are C^r. This set-up is necessary in order to assure that r-normal hyperbolicity is a C^1-open condition.

The *center bolicity* of f is the ratio

$$b = \frac{\|T^c f\|}{m(T^c f)}.$$

The map f is said to be *center bunched* if b is close to 1. More precisely (see §4 of [PS2]; our notation is different): let $0 < \gamma, \nu, \lambda < 1$ be defined by $\gamma := \min\{m(T^c f), \|T^c f\|^{-1}\}$, $\nu := \min\{m(T^s f), \|T^u f\|^{-1}\}$, $\lambda := \max\{\|T^u f^{-1}\|, \|T^s f\|\}$ and set $q := \log_\lambda \nu \geq 1$; then f is center bunched if

$$\frac{\log \gamma}{\log \lambda} < \frac{3 + 2q - \sqrt{(3+2q)^2 - 8}}{4}.$$

In particular, $\nu < \lambda < \gamma$ and the spectra of $T^s f$, $T^c f$ and $T^u f$ are contained in the annuli corresponding to the intervals $[\nu, \lambda]$, $[\gamma, \gamma^{-1}]$ respectively $[\lambda^{-1}, \nu^{-1}]$. This is a stricter condition that the "relative" partial hyperbolicity introduced in (2.1).

By Theorem 6.1 in [HPS], r-normal hyperbolicity is an open property in $\mathrm{Diff}^r(X)$. The property of being center bunched is preserved by C^1-small perturbations. By Proposition 2.3 of [PS1], dynamical coherence is stable under C^1-small perturbations, provided the center lamination is plaque expansive. Plaque expansiveness is a technical condition on an f-invariant lamination. Intuitively it means that separate leaves of the lamination eventually diverge to a fixed distance under either forward or backward f-iterations (see §7 of [HPS] or Appendix 2 for more details). For Anosov diffeomorphisms, i.e., when the center lamination consists of points, this property reduces to expansivity. By Theorem 7.2 of [HPS], the center lamination is plaque expansive if it is a C^1-foliation.

Definition. If $\mathcal{F}$ is a foliation of a Riemannian manifold M and $\mathcal{F}(x)$ is the leaf passing through $x \in M$, denote by $\mathcal{F}_\delta(x)$ the connected component of the intersection of $\mathcal{F}(x)$ and the open ball of radius δ around x.

The following *local product structure property* follows immediately from compactness and transversality for any partially hyperbolic map that has the distributions $E^s \oplus E^c$ and $E^u \oplus E^c$ tangent to the continuous foliation W^{cs} and W^{cu} with C^1-leaves:

There exist constants $\varepsilon > 0, \delta > 0$ and $K > 1$ such that for any $x, y \in X$ with $\mathrm{dist}(x, y) < \varepsilon$, there is a unique point z_1 in $W^s_\delta(x) \cap W^{cu}_\delta(y)$, and a unique point z_2 in $W^u_\delta(x) \cap W^{cs}_\delta(y)$. Moreover:

$$\max\{\mathrm{dist}(x, z_1), \mathrm{dist}(y, z_1)\} < K \,\mathrm{dist}(x, y),$$
$$\max\{\mathrm{dist}(x, z_2), \mathrm{dist}(y, z_2)\} < K \,\mathrm{dist}(x, y);$$

— the submanifold pairs $W^{cs}_\delta(x)$ and $W^{cu}_\delta(y)$, respectively $W^{cs}_\delta(y)$ and $W^{cu}_\delta(x)$, intersect transversally;

— $W^{cs}_\delta(x) \cap W^{cu}_\delta(x) = W^c_\delta(x)$ and $W^{cs}_\delta(y) \cap W^{cu}_\delta(y) = W^c_\delta(y)$.

By Theorem 6.1 of [HPS], the quantities ε, δ and K are lower-semicontinuous with respect to the C^1-topology on $\mathrm{Diff}^1(X)$.

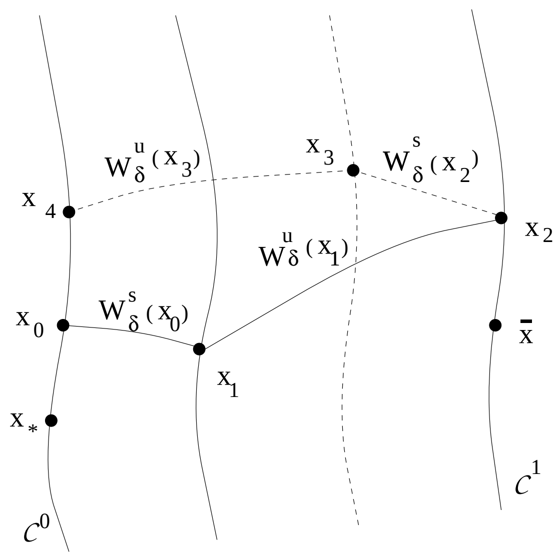
Definition 2.1. Two neutral leaves $\mathcal{C}_0, \mathcal{C}_1$ are called *nearby* if the Hausdorff distance between them is smaller than $\delta' = \delta'(f) := \varepsilon/(5K^3)$.

A point $x \in X$ is *close enough* to a center leaf $\mathcal{C}$ if $\operatorname{dist}(x, \mathcal{C}) < \delta'$.

Definition 2.2. Given a center leaf $\mathcal{C}^0$ and points $x_* \in \mathcal{C}^0$, $\bar{x} \in X$ such that $\operatorname{dist}(x_*, \bar{x}) < \delta'$, we define the holonomy $H_{x_*,\bar{x}} : \mathcal{C}^0_{\delta'}(x_*) \to \mathcal{C}^0$ by $H_{x_*,\bar{x}}(x_0) = x_4$, where $x_0 \in \mathcal{C}^0_{\delta'}(x_*)$ and x_4 is constructed as follows:

$$\begin{aligned}
x_1 :=& W^s_\delta(x_0) \cap W^{cu}_\delta(\bar{x}),\\
x_2 :=& W^u_\delta(x_1) \cap W^{cs}_\delta(\bar{x}) \subset W^u_\delta(x_1) \cap \mathcal{C}^1,\\
x_3 :=& W^s_\delta(x_2) \cap W^{cu}_\delta(x_0),\\
x_4 :=& W^u_\delta(x_3) \cap W^{cs}_\delta(x_0) \subset W^u_\delta(x_3) \cap \mathcal{C}^0.
\end{aligned}$$

If necessary, we will refer to the points x_k in the above construction by $H^{(k)}_{x_*,\bar{x}}(x_0)$, $1 \le k \le 4$. Here $\mathcal{C}^1$ is the center leaf containing $\bar{x}$. The choice of δ' assures that the intersections defining $x_1, \dots, x_4$ have exactly one element.



Note that the pair (x_0, x_4) is (u, s)-accessible.

Remark 2.3. Assume that in the above definition $\mathcal{C}^0$ is connected and one-dimensional, and $x_0 \neq x_4$. Then there is an open neighborhood of x_0 in $\mathcal{C}^0$ whose points are accessible from x_0.

Indeed, by choosing a small continuous path $x_2(t)$, $t \in [0, 1]$ that connects $x_2 = x_2(0)$ to $x_0 = x_2(1)$ and constructing the (u, s)-loops given by $H_{x_0,x_2(t)}(x_0)$, $t \in [0, 1]$, we can access from x_0 at least a closed interval I of $\mathcal{C}^0$ ending at x_0. Since the foliations W^s and W^u are continuous, there is a very short (u, s) loop starting from a point $x_0' \in \mathcal{C}^0$ close to x_0 but outside I and ending in I. Contract this loop to x_0' with the same procedure as above to cover the half-neighborhood of x_0 not contained in I.

§3. Proof of the main results

Remark. By a C^k-*small* diffeomorphism we mean a diffeomorphism that is C^k-close to the identity. The *support* of a diffeomorphism $f : X \to X$ is the closure of the set $\{x \in X \mid f(x) \neq x\}$.

Lemma 3.1. *Assume that f is a normally hyperbolic diffeomorphism of a compact manifold X (hence, its center distribution is integrable). Then, given any point $x \in X$ and any center leaf $\mathcal{C}$, x can be (u,s)-connected to $\mathcal{C}$.*

In particular, if f has a center leaf $\mathcal{C}_0$ such that any two points of $\mathcal{C}_0$ are accessible, then the whole manifold is accessible.

Proof. Denote by $\mathcal{C}_x$ the set of center leaves that can be accessed from x, and let $\mathcal{D}_x$ be the set of points in X that lie on one of the center leafs in $\mathcal{C}_x$. We will show that $\mathcal{D}_x$ is both open and closed, hence it is equal to X.

Openness follows from local product structure property. To show that it is also closed, let $\mathcal{C} \notin \mathcal{C}_x$. Then, again by the local product structure, there is an ε-neighborhood of $\mathcal{C}$ that does not intersect $\mathcal{D}_x$. This shows that the complement of $\mathcal{D}_x$ is open.

For the last statement, notice that given $x, y \in X$ there are $x', y' \in \mathcal{C}_0$ such that the pairs (x, x') and (y, y') are accessible. By the hypothesis, the pair (x', y') is also accessible. □

Theorem 3.2. *Let f be a 1-normally hyperbolic diffeomorphism whose central leaves are 1-dimensional. Let $\mathcal{C}_0$ be a compact periodic central leaf of f, $f^p(\mathcal{C}_0) = \mathcal{C}_0$. Set $F = f^p$.*

Assume that:

1. *$F \restriction_{\mathcal{C}_0}$ has rational rotation number;*
2. *all periodic points of $F \restriction_{\mathcal{C}_0}$ are transverse (hence finitely many only; denote them by $y_i, i = 1, \dots, K$);*
3. *there are points $\bar{y}_i$ close to y_i (i.e., $\operatorname{dist}(y_i, \bar{y}_i) < \delta'(f)$) such that*

$$H_{y_i,\bar{y}_i}(y_i) \neq y_i, \quad \textit{for all} \quad i = 1, \dots, K.$$

Then any two points of $\mathcal{C}_0$ are (u,s)-accessible. Hence, by Lemma 3.1, f is accessible.

Remark. Condition (3) can be replaced by the fact that the local (u,s)-holonomy is non-trivial at each point y_i: there is a point $y_i' \in \mathcal{C}_0$, $y_i' \neq y_i$, which is (u,s)-accessible from y_i through a path that is null-homotopic (mod $\mathcal{C}_0$).

Proof. If the map f satisfies (3), then it follows from Remark 2.3 that there are open connected neighborhoods $U_i \subset \mathcal{C}_0$ of the points y_i's such that each U_i is accessible. Denote

$$\mathcal{U} = \cup_{i=1}^n U_i.$$

Since the periodic points of $F \restriction_{\mathcal{C}_0}$ are transverse, there is a positive integer k_0 such that:

$$\cup_{i=-k_0}^{k_0} F^i(\mathcal{U}) = \mathcal{C}_0.$$

Because the stable and unstable foliations are invariant under f, it follows that if a set A is accessible then the set $f(A)$ is accessible as well. Moreover, if the sets A and B are accessible and $A \cap B \neq \emptyset$, then $A \cup B$ is accessible. Therefore $\mathcal{C}_0$ is accessible. □

The properties (1), (2) and (3) that appear in the statement of Theorem 3.2 are stable under small C^1 perturbations, hence, in order to prove Theorem 1.2 it is enough to show the following:

Theorem 3.3. *Fix $r \geq 1$. Let f be an r-normally hyperbolic diffeomorphism with 1-dimensional center distribution that has two nearby compact periodic center leaves. Then there are arbitrarily C^r-small diffeomorphisms $g \in \mathrm{Diff}^r(X)$ such that $g \circ f$ satisfies the conditions of Theorem 3.2. If a smooth volume form μ is given, one can choose g to preserve μ.*

More precisely, we need the following conditions on the center leaves of f:

— there is a periodic compact 1-dimensional center leaf $\mathcal{C}_0$, and

— each point of $\mathcal{C}_0$ is close enough to a periodic compact center leaf that is disjoint from $\mathcal{C}_0$ (by compactness, finitely many such leaves fulfill this condition for all the points of $\mathcal{C}_0$).

We prove Theorem 3.3 after a few preliminary lemmas. We will assume that a smooth volume form μ is given. The strategy is as follows: we do all the perturbations by composing the original map by time-t maps of certain volume preserving C^r-flows. These flows are constructed in Lemma 3.4. The effect of these perturbations on the center leaves is discussed in Lemma 3.5. Lemmas 3.6 and 3.7 show how to obtain conditions (1) and (2) of the Theorem 3.2. After presenting these Lemmas we describe the remaining part of the proof.

Lemma 3.4. *Let M be a C^∞ manifold and μ a smooth volume form on M. Assume $N \subset M$ is a compact C^r submanifold of codimension at least one, possible with boundary ∂N. Let U be an open neighborhood of N.*

Let X be a C^{r-1} vector field on N which vanishes in a neighborhood of ∂N. Fix an open set $\Omega \subset N$ such that $\mathrm{supp}(X) \subset \Omega \subset \overline{\Omega} \subset N \setminus \partial N$ and a neighborhood $\mathcal{V}$ of X in $\mathcal{X}^{r-1}(N)$, the space of C^{r-1} vector fields on N.

Then there is a continuous path of volume preserving diffeomorphisms $\phi : \mathbb{R} \to \mathrm{Diff}^r_\mu(M)$ given by a non-autonomous flow, such that:

- $\phi_0 = \mathrm{Id}_M$;
- $\mathrm{supp}(\phi_t) \subset U$ *and* $\phi_t(N) = N$ *for* $t \in \mathbb{R}$;
- $X_0 \restriction_N \in \mathcal{V}$ *and* $\mathrm{supp}(X_0 \restriction_N) \subset \Omega$, *where* $X_0 := \dfrac{d}{dt}\phi_t \restriction_{t=0}$.

Remark. In some cases (e.g., in the situation described by Corollary 1.7), we can realize ϕ as the flow of an autonomous vector field.

Proof. Denote $\dim(N) = n$ and $\dim(M) = m$. For convenience we assume that M is endowed with a smooth metric.

Due to the boundary of N, we have to consider open *half-balls* as well, i.e. sets $\{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid \|\mathbf{x}\| < r, x_1 \geq 0\}$.

Choose a finite cover of N by balls $\{U_\alpha\}$ in M such that $U_\alpha \cap N$ are open (half-) balls, $\overline{U_\alpha} \subset U'_\alpha$ and $\chi'_\alpha : U'_\alpha \to V'_\alpha \subset \mathbb{R}^m$ are smooth coordinate maps of M. By applying the Theorem of Moser, [M], to the compactification of $\chi'_\alpha(U'_\alpha)$ we may assume that $(\chi'_\alpha)_* \mu$ is the Lebesgue measure on $V_\alpha := \chi'_\alpha(U_\alpha)$. After refining the original cover and composing the χ'_α's by volume preserving affine transformations, we may assume that for each α the set $\chi'_\alpha(N \cap U_\alpha)$ is given in $\mathbb{R}^m$ by the graph of a C^r function $f_\alpha : D'_\alpha \to B_\alpha$ where $D'_\alpha \subset D_\alpha \subset \mathbb{R}^n$, $B_\alpha \subset \mathbb{R}^{m-n}$, D_α and B_α are open balls, D'_α is either D_α or a half-ball of it, and $D_\alpha \times B_\alpha \subset \chi'_\alpha(U \cap U_\alpha)$. Compose further χ'_α by the volume preserving C^r-map

$(x,y) \in D_\alpha \times B_\alpha \mapsto (x, y - \widetilde{f}(x)) \in \mathbb{R}^n \times \mathbb{R}^{m-n}$, where $\widetilde{f} : D_\alpha \to B_\alpha$ is a C^r-extension of f. We conclude the following:

There is $\varepsilon > 0$ and a finite collection of C^r coordinate maps $\chi : U_\alpha \to V_\alpha$ of M such that:

- $N \subset \cup_\alpha U_\alpha \subset U$;
- $(\chi_\alpha)_* \mu$ is the Lebesgue measure on $\chi_\alpha(U_\alpha) \subset \mathbb{R}^m$;
- $\chi_\alpha(N \cap U_\alpha) = D'_\alpha \times \{0\} \subset D_\alpha \times \mathrm{Ball}_\varepsilon^{m-n}(0) \subset \chi_\alpha(U_\alpha) \subset \mathbb{R}^m$ with $D_\alpha \subset \mathbb{R}^n$ an open ball, where D'_α is equal to either D_α or a half-ball of it.

Choose on M a smooth partition of the unity $\{\psi_\alpha\}$ subordinated to $\{U_\alpha\}$ and let $X_\alpha := \psi_\alpha X \in \mathcal{X}^{r-1}(N)$, $v_\alpha := (\chi_\alpha)_* (X_\alpha)$. Then $v_\alpha \in \mathcal{X}^{r-1}(D_\alpha)$ because X vanished near ∂N. By standard approximation arguments (see, e.g., [H]), one can find C^r vector fields $\widetilde{v}_\alpha \in \mathcal{X}^r(D_\alpha)$ such that $\mathrm{supp}(\widetilde{v}_\alpha)$ is compact in D'_α, $\sum_\alpha \chi^*_\alpha(\widetilde{v}_\alpha) \in \mathcal{V}$ and $\mathrm{supp}(\sum_\alpha \chi^*_\alpha(\widetilde{v}_\alpha)) \subset \Omega$.

We extend each $\widetilde{v}_\alpha$ to a gradient-free compactly supported vector field $\widetilde{w}_\alpha \in \mathcal{X}^r(D_\alpha \times \mathrm{Ball}_\varepsilon^{m-n}(0))$ as follows: denote the coordinates of $\mathbb{R}^n$ by $x_1, \ldots, x_n$, those of $\mathbb{R}^{m-n}$ by $y_1, \ldots, y_{m-n}$, and the components of $\widetilde{v}_\alpha$ by $\widetilde{v}^1_\alpha, \ldots, \widetilde{v}^n_\alpha$. Define

$$\widetilde{w}_\alpha(x,y) := (\psi(y_1) \cdot \widetilde{v}_\alpha(x), Y(x,y), 0, \ldots, 0)\,,$$

where

- $\psi \in C_c^\infty((-\varepsilon, \varepsilon), \mathbb{R})$;
- $\psi(0) = 1$;
- $\displaystyle\int_0^{\pm\varepsilon} \psi(s)\, ds = 0$;
- $\displaystyle Y(x,y) := -\sum_{i=1}^n \frac{\partial \widetilde{v}^i_\alpha}{\partial x_i} \cdot \int_0^{y_1} \psi(s)\, ds.$

Let $\phi^\alpha : \mathbb{R} \to \mathrm{Diff}^r_{\mathrm{Leb}}(\chi_\alpha(U_\alpha))$ be the flow generated by $\widetilde{w}_\alpha$. Then $\phi : \mathbb{R} \to \mathrm{Diff}^r_\mu(M)$ given by

$$\phi_t := \prod_\alpha \left(\chi_\alpha^{-1} \circ \phi^\alpha_t \circ \chi_\alpha\right)$$

satisfies the conditions of the Lemma (for the above composition fix any order of the indexes α). $\square$

Lemma 3.5. *Let f be an r-normally hyperbolic diffeomorphism on the compact manifold X and $\mathcal{C}_0$ be a center leaf of f. If $g \in \mathrm{Diff}^r(X)$ is C^1-close to f and $f^n(\mathcal{C}_0) = g^n(\mathcal{C}_0)$ for all $n \in \mathbb{Z}$ then $\mathcal{C}_0$ is a center leaf of g as well.*

Proof. This follows from the characterization of the center leafs of g given in Theorem 6.8 of [HPS]: it roughly says that to each center leaf $\mathcal{C}$ of f there corresponds a center leaf $\mathcal{C}'$ of g, uniquely determined by the fact that $f^n(\mathcal{C})$ and $g^n(\mathcal{C}')$ stay close to each other for all $n \in \mathbb{Z}$. $\square$

We say that the rotation number of a diffeomorphism $\phi \in \mathrm{Diff}^1(S^1)$ is stable if all diffeomorphisms C^1-close to ϕ have the same rotation number.

Lemma 3.6. *Let X be a compact manifold with a smooth volume μ. Given any r-normally hyperbolic map $f \in \mathrm{Diff}^r(X)$ with a p-periodic 1-dimensional compact center leaf $\mathcal{C}_0$, $f^p(\mathcal{C}_0) = \mathcal{C}_0$, there are arbitrarily small diffeomorphisms $g \in \mathrm{Diff}^r_\mu(X)$ such that $g(\mathcal{C}_0) = \mathcal{C}_0$, $\mathcal{C}_0$ is a p-periodic center leaf of $g \circ f$ and the rotation number of $(g \circ f)^p$ on $\mathcal{C}_0$ is rational and stable.*

Proof. Fix a C^r-diffeomorphism $\Phi : S^1 \to \mathcal{C}_0$ and pick a tubular neighborhood U of $\mathcal{C}_0$ such that U does not intersect $\{f^k(\mathcal{C}_0) \mid k = 1, \ldots, p-1\}$.

In view of Lemma 3.5, if $g \in \mathrm{Diff}^r_\mu(X)$ is small enough, $g(\mathcal{C}_0) = \mathcal{C}_0$ and $\mathrm{supp}(g) \subset U$ then $\mathcal{C}_0$ is a center leaf of $g \circ f$ (and, obviously, p-periodic).

Notice first that if $f^p \restriction \mathcal{C}_0$ has irrational rotation number then by an arbitrarily small C^r-perturbation g as above we can achieve that $(g \circ f)^p$ have rational rotation number on $\mathcal{C}_0$. Indeed, by Lemma 3.4, there is a volume preserving C^r-flow $\{\phi_t\}$ supported in U and preserving $\mathcal{C}_0$ with the property that it does not have any fixed point on $\mathcal{C}_0$. Then for small t the circle diffeomorphisms $\Phi^{-1}(\phi_t \circ f)^p\Phi$ and $\Phi^{-1}f^p\Phi$ are ordered in the sense that their "compatible" lifts to $\mathbb{R}$ do not intersect (see [KH], Definition 11.1.7; note that $\Phi^{-1}(\phi_t \circ f)^p\Phi = \Phi^{-1}(\phi_t \circ f^p)\Phi)$. Therefore, by Proposition 11.1.8 of [KH],

$$\rho(\Phi^{-1}(\phi_t \circ f)^p\Phi) \neq \rho(\Phi^{-1}f^p\Phi)$$

for any small $t \neq 0$. In particular, the rotation number has to take rational values for arbitrarily small values of $t \neq 0$.

It remains to show that the rational rotation number can be made stable. Assume that $\rho(F) = \rho_0$, with $\rho_0 \in \mathbb{Q}$, where $F := \Phi^{-1}(\phi_t \circ f)^p\Phi$. If F is not a rotation then it has at least one semistable periodic point, hence "pushing" $\phi_t \circ f$ in the right direction with a flow around $\mathcal{C}_0$ produces a stable periodic point (see Proposition 11.1.10 in [KH]).

If F is a rotation, let (x_*, y_*) be an interval of S^1 that contains no element of the F-orbit of x_*. Take a flow $\{\psi_s\}$ preserving $\mathcal{C}_0$ such that $\emptyset \neq \mathrm{supp}(\psi_s \restriction_{\mathcal{C}_0}) \subset (x_*, y_*)$. Then $(\psi_s \circ \phi_t \circ f)^p$ has rotation number ρ_0 but is not a rotation, hence we can apply the previous argument. □

Lemma 3.7. *Let $f \in \mathrm{Diff}^r(X)$ be an r-normally hyperbolic diffeomorphism that has a p-periodic 1-dimensional compact leaf $\mathcal{C}_0$. Assume that $f^p \restriction_{\mathcal{C}_0}$ has stable rational rotation number ρ_0. Then there are arbitrarily small diffeomorphisms $g \in \mathrm{Diff}^r_\mu(X)$, such that $\mathcal{C}_0$ is a p-periodic leaf of $g \circ f$, and $(g \circ f)^p \restriction_{\mathcal{C}_0}$ has rotation number ρ_0 and only transverse periodic points.*

Proof. Fix a tubular neighborhood U of $\mathcal{C}_0$ such that U does not intersect $\{f^k(\mathcal{C}_0) \mid k = 1, \ldots, p-1\}$. Use Lemma 3.4 to construct a volume preserving flow $\{\phi_t\}$ that preserves $\mathcal{C}_0$, is supported in U, and is nonsingular for $|t| < \varepsilon$. We will obtain the desired perturbation by considering $\phi_t \circ f$ for small values of t.

In view of Lemma 3.5 and our hypothesis, the only part of the conclusion that has to be checked is that one can make all periodic points of $(\phi_t \circ f)^p \restriction_{\mathcal{C}_0}$ transverse. For this we use the following consequence of Sard's Theorem (see [H], Theorem 2.7):

Parametric Transversality Theorem. *Let V, M, N be C^r-manifolds without boundary and $A \subset N$ a C^r-submanifold. Let $F : V \to C^r(M, N)$ be a map that satisfies the following conditions:*

(i) *the evaluation map $F^{ev} : V \times M \to N$, $(v, x) \mapsto F_v(x) = F(v, x)$ is C^r;*
(ii) *F^{ev} is transversal to A;*
(iii) $r > \max\{0, \dim M + \dim A - \dim N\}$.

Then the set $Trans(F; A) = \{v \in V \mid F_v$ is transversal to $A\}$ is dense.

We identify $\mathcal{C}_0$ with S^1, but for simplicity do not write this identification explicitly. Assume that $f^p \restriction_{\mathcal{C}_0}$ has a periodic point of period q.

Let $M = \mathcal{C}_0$, $N = \mathcal{C}_0 \times \mathcal{C}_0$, $A = \{(x, x) \mid x \in \mathcal{C}_0\}$, $V = (-\varepsilon, \varepsilon)$ and

$$F(t) = [x \in M \mapsto (x, (\phi_t \circ f^p)^q(x)) \in N]$$

(note that $(\phi_t \circ f)^p \restriction_{\mathcal{C}_0} = (\phi_t \circ f^p) \restriction_{\mathcal{C}_0}$).

Condition (i) is obvious and condition (iii) holds for $r > 0$. Note that ϕ_t is a diffeomorphism for any t and coincides with the identity for $t = 0$, hence the derivative $D_x(\phi_t \restriction_{\mathcal{C}_0})$ is positive for $t \in (-\varepsilon, \varepsilon)$. A straightforward computation shows that $\frac{\partial}{\partial t}(\phi_t \circ f^p)^q \restriction_x \neq 0$ for any $t \in V$ and $x \in \mathcal{C}_0$, because the partial derivative is the sum of q terms, each having the same sign as the vector field. Hence the vector $\frac{\partial}{\partial t} F^{ev}$ and the tangent space of A span the tangent space of N at each point in the intersection of the image of F^{ev} with A. Therefore condition (ii) is true as well. It follows from the Transversality Theorem that there are values of $t \in (-\varepsilon, \varepsilon)$ arbitrarily close to zero for which $(\phi_t \circ f)^p \restriction_{\mathcal{C}_0}$ has only transverse periodic points. □

The last ingredient is provided by the following:

Lemma 3.8. *Let $f \in \mathrm{Diff}^r(X)$ be a partially hyperbolic diffeomorphism as in Theorem 3.3, having two disjoint periodic compact center leaves $\mathcal{C}_0$ and $\mathcal{C}_1$ of period dividing p. Let K be a compact subset of X that is f-invariant and does not intersect either $\mathcal{C}_0$ or $\mathcal{C}_1$. Given two close enough points $x_* \in \mathcal{C}_0$, $\bar{x} \in \mathcal{C}_1$, there is an arbitrarily C^r-small diffeomorphism $g \in \mathrm{Diff}^r_\mu(X)$ such that $\widetilde{f} := g \circ f$ has $\mathcal{C}_0$ and $\mathcal{C}_1$ as periodic center leaves, $\widetilde{f}^p \restriction_{\mathcal{C}_0} = f^p \restriction_{\mathcal{C}_0}$, $\widetilde{f}^p \restriction_K = f^p \restriction_K$ and*

$$H_{y_*, \bar{y}}(y_*) \neq \widetilde{H}_{y_*, \bar{y}}(y_*),$$

where $H_{y_*, \bar{y}}$ is the f-holonomy and $\widetilde{H}_{y_*, \bar{y}}$ is the $\widetilde{f}$-holonomy.

Remark. In our case K will consist of the f-trajectories of a finite number of periodic compact center leaves. By Lemma 3.5, center leaves of f that are contained in K remain center leaves for $\widetilde{f}$ as well.

Proof. Let $H_{y_*, \bar{y}} : V \to \mathcal{C}_0$ be the f-holonomy, where V is an open subset of $\mathcal{C}_0$ containing y_*. Given $y \in V$, denote $y^{(k)} := H^{(k)}_{y_*, \bar{y}}(y)$ for $1 \leq k \leq 4$ (recall the notations introduced in Definition 2.2). Let $N := \{y^{(3)} \mid y \in V\} \subset W^{cu}(y_*^{(4)}) \cap W^{cs}(y_*^{(2)})$ and $N^k := f^{-k}(N)$. Note that $y_*^{(3)} \in N$ and by transversality N is a C^r-manifold.

Since N^{np} approaches $\mathcal{C}_0$ as $n \to \infty$, there is a positive multiple q of p such that $N^q \cap K = \emptyset$ and

$$N^q \cap f^\ell(W^{cs}_\delta(\mathcal{C}_1)) = \emptyset \quad \text{for} \quad \ell = 0, 1, \dots, p-1,$$

and thus for all $\ell \geq 0$, because $f^p(W^{cs}_\delta(\mathcal{C}_1)) \subset W^{cs}_\delta(\mathcal{C}_1)$. Moreover, for $k, \ell \in \mathbb{Z}$,

$$N^k \cap f^\ell(W^{cs}_\delta(\mathcal{C}_0)) = \emptyset, \qquad N^k \cap f^\ell(W^{cu}_\delta(\mathcal{C}_1)) = \emptyset,$$

as one can see by considering the ω-, respectively α-limit sets under f^p.

Since $y^{(3)}_*$ is backward asymptotic to the f-orbit of $\mathcal{C}_0$, we may shrink V (and thus N) so that

$$N^k \cap N^q = \emptyset \quad \text{for} \quad k \geq 0,\ k \neq q.$$

Then there is an open neighborhood U of N^q which is disjoint from K, N^k for $k \geq 0$, $k \neq q$, and $f^\ell(W^{cs}_\delta(\mathcal{C}_0))$, $f^\ell(W^{cs}_\delta(\mathcal{C}_1))$, $f^{-\ell}(W^{cu}_\delta(\mathcal{C}_1))$, for $\ell = 0, \dots, p-1$.

Let $g \in \mathrm{Diff}^r(X)$ be a small diffeomorphism such that $g(N^q) = N^q$ and $\mathrm{supp}(g) \subset U$. Let $\widetilde{f} := g \circ f$ and denote the stable and unstable foliations of $\widetilde{f}$ by $\widetilde{W}^s$, respectively $\widetilde{W}^u$.

By Lemma 3.5, $\mathcal{C}_0$ and $\mathcal{C}_1$ are center leafs of $\widetilde{f}$ as well, hence we can consider the $\widetilde{f}$-holonomy $\widetilde{H}_{y_*,\bar{y}} : V \to \mathcal{C}_0$. Note that $N^{np} \subset \widetilde{W}^{cu}(\mathcal{C}_0) \cap W^{cu}(\mathcal{C}_0)$ for all $n \geq 0$, because N^{np} is backward asymptotic to $\mathcal{C}_0$ under both $\widetilde{f}^p$ and f^p.

The desired conclusion is a consequence of the following formula describing how the holonomy changes:

$$\widetilde{H}_{y_*,\bar{y}} = f^q \circ \Phi \circ g^{-1} \circ \Phi^{-1} \circ f^{-q} \circ H_{y_*,\bar{y}}, \tag{3.1}$$

where $\Phi : N^q \to f^{-q}(H_{y_*,\bar{y}}(V)) \subset \mathcal{C}_0$ is the holonomy along the unstable foliation of f within $W^{cu}(\mathcal{C}_0)$.

Indeed, pick $y \in V$ and let $\widetilde{y}^{(k)} := \widetilde{H}^{(k)}_{y_*,\bar{y}}(y)$ for $1 \leq k \leq 4$ (the notations correspond to those of Definition 2.2, applied for $\widetilde{f}$). Since f^n coincides with $\widetilde{f}^n$ on $W^{cs}_\delta(y)$ and $W^{cs}_\delta(y^{(2)})$ for all $n \geq 0$, $\widetilde{W}^s_\delta(y) = W^s_\delta(y)$ and $\widetilde{W}^s_\delta(y^{(2)}) = W^s_\delta(y^{(2)})$. Similarly, $\widetilde{W}^u_\delta(y^{(2)}) = W^u_\delta(y^{(2)})$. We conclude that $\widetilde{y}^{(k)} = y^{(k)}$ for $y \in V$ and $1 \leq k \leq 3$.

It remains to consider the transition from N to $\mathcal{C}_0$. Denote by H, respectively $\widetilde{H}$ the holonomy from N to $\mathcal{C}_0$ along W^u within $W^{cu}(\mathcal{C}_0)$, respectively along $\widetilde{W}^u$ within $\widetilde{W}^{cu}(\mathcal{C}_0)$. The functions f^{-np} and $\widetilde{f}^{-np}$ coincide on N^{q+p} for all $n \geq 0$, therefore the holonomy Φ_0 along the unstable leaves from N^{q+p} to $\mathcal{C}_0$ is the same for both f and $\widetilde{f}$ (recall that the points of the local unstable leaf of $x \in X$ are characterized by the fact their backward trajectory does not stray away from that of x). By the invariance of the unstable foliations we obtain that

$$f^{q+p} \circ \Phi_0 = H \circ f^{q+p}, \quad \text{respectively} \quad \widetilde{f}^{q+p} \circ \Phi_0 = \widetilde{H} \circ \widetilde{f}^{q+p},$$

both relations being considered on N^{q+p}. Since $\widetilde{f}^{q+p} \restriction_{N^{q+p}} = f^q \circ g \circ f^p \restriction_{N^{q+p}}$, we obtain that

$$\widetilde{H} \restriction_N = f^{q+p} \circ \Phi_0 \circ f^{-p} \circ g^{-1} \circ f^p \circ \Phi_0^{-1} \circ f^{-q-p} \circ H \restriction_N .$$

By the same invariance, $f^p \circ \Phi_0 \circ f^{-p} = \Phi$, which implies (3.1).

By Lemma 3.4, there is a flow $\phi_t \in \mathrm{Diff}^r_\mu(X)$ that is supported in U, preserves N^q and does not fix the point $\Phi^{-1} \circ f^{-q}(y_*^{(4)})$ for small $t \neq 0$. Therefore, there is $t_0 > 0$ such that

$$\widetilde{H}_{y_*,\bar{y}}(y_*) \neq H_{y_*,\bar{y}}(y_*),$$

provided $g = \phi_t$ and $0 < |t| < t_0$. □

Proof of Theorem 3.3. Let $f \in \mathrm{Diff}^r(X)$ be as given in the theorem, with $\mathcal{C}_0$ a compact periodic center leaf.

By the Lemmas 3.6 and 3.7, we can find arbitrarily small diffeomorphisms $g \in \mathrm{Diff}^r_\mu(X)$ such that $g \circ f$ satisfies conditions (1) and (2) of Theorem 3.2. Thus, we may assume without loss of generality that f itself satisfies conditions (1) and (2). We will construct a small perturbation $\widetilde{f}$ of f which induces on $\mathcal{C}_0$ the same map as f and satisfies condition (3) as well.

Let p_0 be the period of $\mathcal{C}_0$. Denote by y_i, $1 \leq i \leq K$, the periodic points of f^{p_0} on $\mathcal{C}_0$. Let $\bar{y}_i$ be points close enough to y_i situated on periodic compact leaves $\mathcal{C}_i$ and $H_i := H_{y_i,\bar{y}_i} : V_i \to \mathcal{C}_0$ the corresponding f-holonomies (we are not going to write explicitly the dependence of the holonomy on the function unless it is not clear from the context). We label the points y_i as "bad" or "good", depending on whether they satisfy condition (3):

$$B := \{y_i \mid H_i(y_i) = y_i, 1 \leq i \leq K\}, \qquad G := \{y_i \mid H_i(y_i) \neq y_i, 1 \leq i \leq K\},$$

We use Lemma 3.8 to do successive perturbations which move points from B to G. During these perturbations $\mathcal{C}_0$ and $\mathcal{C}_i$ remain periodic center leaves and $f^{p_0} \restriction_{\mathcal{C}_0}$ is not affected, hence the set of periodic points on $\mathcal{C}_0$ does not change and conditions (1) and (2) are still fulfilled. Due to the continuity of the stable and unstable foliations with respect to changes in the diffeomorphism, small perturbations will never take a point from G to B. Therefore, after a finite number of perturbations all points y_i will be in G, as desired. □

§4. Appendix 1

Let M and N be compact manifolds.

Consider on $\mathrm{Homeo}(N)$ the distance $\mathbf{d}_N(g,h) := \sup_{y\in N} \mathrm{dist}_N(g(y), h(y))$, where $g, h \in \mathrm{Homeo}(N)$. It has the following properties: if $g, h, u, v \in \mathrm{Homeo}(N)$ then

$$\mathbf{d}_N(gu, hu) = \mathbf{d}_N(g,h);$$
$$\mathbf{d}_N(ug, uh) \leq \|u\|_{\mathrm{Lip}} \mathbf{d}_N(g,h),$$

hence

$$\mathbf{d}_N(gh, uv) \leq \mathbf{d}_N(gh, uh) + \mathbf{d}_N(uh, uv) \leq \mathbf{d}_N(g,u) + \|u\|_{\mathrm{Lip}} \mathbf{d}_N(h,v); \tag{4.1}$$
$$\mathbf{d}_N(g^{-1}, h^{-1}) = \mathbf{d}_N(g^{-1}h, h^{-1}h) = \mathbf{d}_N(g^{-1}h, g^{-1}g) \leq \|g^{-1}\|_{\mathrm{Lip}} \mathbf{d}_N(g,h). \tag{4.2}$$

Fix a C^1 Anosov diffeomorphism $A : M \to M$. There is $\lambda \in (0,1)$, and a splitting $TM = E^s \oplus E^u$ such that with respect to an adapted Riemannian metric on M

$$\|TAv\| \leq \lambda \|v\|, \quad v \in E^s,$$
$$\|TA^{-1}v\| \leq \lambda \|v\|, \quad v \in E^u.$$

Definition. A map $\beta : \mathbb{Z} \times M \to \mathrm{Diff}^1(N)$ is called a *cocycle over* A if it satisfies

$$\beta(m+n, x) = \beta(m, A^n x) \circ \beta(n, x) \tag{4.3}$$

for $m, n \in \mathbb{Z}$, and $x \in M$.

Given $\theta \in (0,1]$, the cocycle β is said to be *θ-Hölder* if there is a constant $C_H > 0$ such that

$$\mathbf{d}_N(\beta(1,x), \beta(1,y)) \leq C_H \operatorname{dist}_M(x,y)^\theta \qquad x, y \in M, \tag{4.4}$$

The smallest value C_H that can be used in the above formula is called the *θ-Hölder (semi) norm* of β, denoted by $\|\beta(1,\cdot)\|_\theta$.

The cocycle β is said to be *θ-close to the identity* if

$$\|T\beta(1,\cdot)^{\pm 1}\| \lambda^\theta < 1. \tag{4.5}$$

In view of (4.2), the conditions (4.4) and (4.5) imply that $\beta(1,\cdot)^{-1}$ is θ-Hölder as well, $\|\beta(1,\cdot)^{-1}\|_\theta \leq \lambda^{-\theta} \|\beta(1,\cdot)\|_\theta$.

A cocycle $\beta : \mathbb{Z} \times M \to \mathrm{Diff}^1(N)$ determines a skew-product $f_\beta : M \times N \to M \times N$ given by

$$f_\beta(x,y) := (Ax, \beta(1,x)(y)).$$

For $0 < \delta < 1$, $0 < H < \infty$ denote by $\mathrm{Sk}_{H,\delta}(A,N)$ the set of skew-products over A given by cocycles $\beta : \mathbb{Z} \times M \to \mathrm{Diff}^1(N)$ that are θ-Hölder for some $\theta \in (0,1]$ and satisfy $\|\beta(1,\cdot)\|_\theta < H$, $\|T\beta(1,\cdot)^{\pm 1}\|\lambda^\theta < \delta$. It follows from formula (4.3) that a cocycle β is determined by the map $\beta(1,\cdot) : M \to \mathrm{Diff}^1(N)$, hence we can see $\mathrm{Sk}_{H,\delta}(A,N)$ as a subset of $\mathrm{Maps}(M, \mathrm{Diff}^1(N))$. Endow the space of maps from M to $\mathrm{Diff}^1(N)$ with the topology of uniform convergence and consider on $\mathrm{Sk}_{H,\delta}(A,N)$ the induced topology.

For maps in $\mathrm{Sk}_{H,\delta}(A,N)$ one can construct the equivalent of the stable and unstable foliations of partially hyperbolic C^1-maps. Since the existence of these foliations for Hölder skew-products does not follow from the standard theory of partially hyperbolic diffeomorphisms, we include a proof of this fact in Theorem 4.3. Using these foliations, one can speak of accessibility. The stable and unstable foliations of the skew-product are contracting, respectively expanding, and the leaves of the foliations depend continuously on the cocycle β. Therefore, the proof of Theorem 1.2 can be applied to yield:

Theorem 4.1. *In* $\mathrm{Sk}_{H,\delta}(A,S^1)$ *the accessible maps contain an open and dense set.* □

Another motivation for Theorem 4.3 is a generalization of a result of Brin (see [B]) about topologically transitive C^2 skew-products. Indeed, if G is a compact connected Lie group, then the set of translations by elements in G can be embedded in $\mathrm{Diff}^1(G)$ as Haar measure preserving transformations. Denote by $\mathrm{Sk}^G(A)$ the set of skew-products f_β over A given by Hölder cocycles $\beta : \mathbb{Z} \times M \to G$. Note that any element f_β in $\mathrm{Sk}^G(A)$ belongs to $\mathrm{Sk}_{H,\delta}(A, G)$, for some H, δ. Denote $\mathrm{Sk}^G_{H,\delta}(A) = \mathrm{Sk}_{H,\delta}(A, G) \cap \mathrm{Sk}^G(A)$. Any element in $\mathrm{Sk}^G_{H,\delta}(A)$ has stable and unstable foliations satisfying all the properties in Theorem 4.3. Using this fact, one can follow [B] to prove:

Theorem 4.2. *Let $A : M \to M$ be a topologically transitive C^1 Anosov diffeomorphism and G a compact connected Lie group. Then $\mathrm{Sk}^G_{H,\delta}(A)$ contains an open dense set of topologically transitive transformations.* □

Indeed, according to [B], one has to check that a generic map is recurrent, and has the accessibility property. Since A is topologically transitive, it has a dense set of periodic points. From this follows that any skew-product in $\mathrm{Sk}^G_{H,\delta}(A)$ is recurrent. Generic accessibility can be proved as in [B], using the properties of the stable and unstable foliations listed in Theorem 4.3.

We will discuss in Theorem 4.3 the stable foliation only; for the unstable foliation consider f^{-1} instead of f. In order to simplify the notation, denote $\beta(1, x)$ by $\beta(x)$. In the sequel W^s stands for the stable foliation of the Anosov diffeomorphism $A : M \to M$ and dist_s for the distance induced by the metric of M on a leaf of W^s.

Theorem 4.3. *Let $A : M \to M$ be the C^1 Anosov diffeomorphism considered above and $\beta : \mathbb{Z} \times M \to \mathrm{Diff}^1(N)$ a θ-Hölder cocycle which is θ-close to the identity. Define $\gamma_{x,n} : W^s(x) \to \mathrm{Diff}^1(N) \subset \mathrm{Homeo}(N)$ by*

$$\gamma_{x,n}(t) := \beta(n, t)^{-1}\beta(n, x).$$

Then:

(a) *The sequence $\{\gamma_{x,n}\}_{n\geq 0}$ is pointwise convergent in* $\mathrm{Homeo}(N)$.

(b) *The limit*

$$\gamma_x := \lim_{n\to\infty} \gamma_{x,n} : W^s(x) \to \mathrm{Homeo}(N)$$

is uniformly θ-Hölder from dist_s to $\mathbf{d}_N$.

(c) $\gamma_x(x) = \mathrm{Id}_N$.

(d) *The family of graphs $\mathbf{W}^s(x, y) := \{(t, \gamma_x(t)y) \mid t \in W^s(x)\}$, $x \in M, y \in N$, gives an f-invariant foliation of $M \times N$, where $f \in \mathrm{Homeo}(M \times N)$ is the skew-product*

$$f(x, y) := (Ax, \beta(x)y), \qquad x \in M, y \in N.$$

(e) *If $t \in W^s(x)$ and $\nu > \lambda^\theta$ then*

$$\lim_{n\to\infty} \nu^{-n}\,\mathrm{dist}_{M\times N}(f^n(x, y), f^n(t, \gamma_x(t)(y))) = 0,$$

uniformly in $(x, y) \in M \times N$ and $\{t \in W^s(x) \mid \mathrm{dist}_s(x, t) \leq K\}$, for any $K > 0$.

(f) *The family of maps $\{\gamma_x : W^s(x) \to \mathrm{Homeo}(N) \mid x \in M\}$ is uniquely determined by either of the following conditions:*

(i) *properties (b), (c) and (d);*

(ii) *property (e) for a value ν satisfying $\lambda^\theta < \nu < m(T\beta) = \|T\beta^{-1}\|^{-1}$.*

(g) *The application*

$$\beta : M \to \mathrm{Diff}^1(N) \in \{\beta \mid f_\beta \in \mathrm{Sk}_{H,\delta}(A,N)\} \mapsto \{\gamma_x : W^s(x) \to \mathrm{Homeo}(N)\}_{x\in M}$$

constructed in (a) is continuous from the topology of uniform convergence of maps from M to $\mathrm{Homeo}(N)$ *to the topology of uniform convergence on compact sets of maps from the leafs of W^s to* $\mathrm{Homeo}(N)$.

Proof. We will use (4.1) repeatedly to estimate the distance between two products in $\mathrm{Homeo}(N)$.

Recall (see [P]) that there is a constant $C_s > 1$ such that

$$\mathrm{dist}_s(A^n x, A^n t) \le C_s \lambda^n \,\mathrm{dist}_s(x,t) \tag{4.6}$$

for $n \ge 0$ and $t \in W^s(x)$.

(a) It is enough to show that the sequence $\{\gamma_{x,n}(t)\}_n$ is Cauchy. Let $m \ge n$ be positive integers.

$$\begin{aligned}
&\mathbf{d}_N(\gamma_{x,n}(t), \gamma_{x,m}(t)) \\
&= \mathbf{d}_N(\beta(t)^{-1} \circ \cdots \circ \beta(A^{n-1}t)^{-1} \circ \beta(A^n x)^{-1} \circ \cdots \circ \beta(A^{m-1}x)^{-1} \circ \beta(m,x), \\
&\qquad \beta(t)^{-1} \circ \cdots \circ \beta(A^{n-1}t)^{-1} \circ \beta(A^n t)^{-1} \circ \cdots \circ \beta(A^{m-1}t)^{-1} \circ \beta(m,x)) \\
&\le \sum_{k=n}^{m-1} \|\beta(t)^{-1} \circ \cdots \circ \beta(A^{k-1}t)^{-1}\|_{\mathrm{Lip}} \mathbf{d}_N(\beta(A^k x)^{-1}, \beta(A^k t)^{-1}) \\
&\le C_1 \|T\beta^{-1}\|^n \lambda^{n\theta} \,\mathrm{dist}_s(x,t)^\theta,
\end{aligned} \tag{4.7}$$

where

$$C_1 = \frac{\|\beta(\,\cdot\,)^{-1}\|_\theta}{1 - \|T\beta^{-1}\|\lambda^\theta} \cdot C_s^\theta.$$

(b) Let $s, t \in W^s(x)$, and n be a positive integer. Then:

$$\begin{aligned}
&\mathbf{d}_N(\gamma_{x,n}(s), \gamma_{x,n}(t)) \\
&= \mathbf{d}_N(\beta(s)^{-1} \circ \cdots \circ \beta(A^{n-1}s)^{-1}\beta(n,x), \beta(t)^{-1} \circ \cdots \circ \beta(A^{n-1}t)^{-1}\beta(n,x)) \\
&\le C_2 \sum_{k=0}^{n-1} \|T\beta^{-1}\|^k_{\mathrm{Lip}} \lambda^{k\theta} \,\mathrm{dist}_s(s,t)^\theta \le C_3 \,\mathrm{dist}_s(s,t)^\theta,
\end{aligned}$$

where the constants C_2, C_3 are independent of s, t, m, n, x, β.

(c) This is obvious.

(d) The invariance is equivalent to

$$\beta(n,t)\gamma_x(t) = \gamma_{A^n x}(A^n t)\beta(n,x), \qquad t \in W^s(x)$$

and this follows from the identity

$$\beta(n,t)\gamma_{x,m}(t) = \gamma_{A^n x, m-n}(A^n t)\beta(n,x).$$

(e) One has:

$$f^n(x,y) = (A^n x, \beta(n,x)(y)),$$
$$f^n(t, \gamma_x(t)(y)) = (A^n, \beta(n,t)\gamma_x(t)(y)).$$

In view of (4.6) it only remains to prove that

$$\lim_{n\to\infty} \nu^{-n} \mathbf{d}_N(\beta(n,x), \beta(n,t)\gamma_x(t)) = 0.$$

By (d), (c) and (b)

$$\mathbf{d}_N(\beta(n,x), \beta(n,t)\gamma_x(t)) = \mathbf{d}_N(\beta(n,x), \gamma_{A^n x}(A^n t)\beta(n,x))$$
$$= \mathbf{d}_N(\mathrm{Id}_N, \gamma_{A^n x}(A^n t)) \le C_3 \lambda^{n\theta} \operatorname{dist}_s(x,t)^\theta.$$

(f) Let $\{\omega_x : W^s(x) \to \mathrm{Homeo}(N) \mid x \in M\}$ be another family of functions.

(i) Assume that $\{\omega_x\}$ satisfies (b), (c) and (d). From (d) we conclude that $\omega_x(t) = \beta(n,t)^{-1}\omega_{A^n x}(A^n t)\beta(n,x)$, $t \in W^s(x)$. But then, using also (c) and (b):

$$\mathbf{d}_N(\omega_x(t), \gamma_{x,n}(t)) = \mathbf{d}_N(\beta(n,t)^{-1}\omega_{A^n x}(A^n t)\beta(n,x), \beta(n,t)^{-1}\beta(n,x))$$
$$\le \|\beta(n,t)^{-1}\|_{\mathrm{Lip}} \mathbf{d}_N(\omega_{A^n x}(A^n t), \mathrm{Id}_N) \le C_4 \|T\beta^{-1}\|^n \lambda^{n\theta} \operatorname{dist}_s(x,t),$$

which implies the desired conclusion.

(ii) Assume that $\{\omega_x\}$ satisfies (e). Then

$$\lim_{n\to\infty} \nu^{-n} \mathbf{d}_N(\beta(n,x)\gamma_x(t), \beta(n,t)\omega_x(t)) = 0 \quad \text{for} \quad t \in W^s(x).$$

But

$$\nu^{-n} \mathbf{d}_N(\beta(n,x)\gamma_x(t), \beta(n,t)\omega_x(t)) \ge \nu^{-n} \|\beta(n,x)^{-1}\|_{\mathrm{Lip}}^{-1} \mathbf{d}_N(\gamma_x(t), \omega_x(t)),$$

and the right-hand side converges to zero for $\lambda^\theta < \nu < \|T\beta^{-1}\|^{-1}$ if and only if $\mathbf{d}_N(\gamma_x(t), \omega_x(t)) = 0$.

(g) This is a consequence of the fact that the application $\beta \mapsto \gamma_x$ is the "uniform limit of a sequence of continuous functions".

Indeed, let β and β' be two cocycles whose skew-products are in $\mathrm{Sk}_{H,\delta}(A,N)$. Denote by $\gamma_{x,n}, \gamma_x$ and $\gamma'_{x,n}, \gamma'_x$ the corresponding families considered in (a).

Let $t \in W^s(x)$. The estimate (4.7) implies that

$$\mathbf{d}_N(\gamma_{x,n}(t), \gamma_x(t)) \le \frac{H\lambda^{-1}}{1-\delta} C_s \delta^n \max\{\operatorname{dist}_s(t,x), 1\},$$

and a similar inequality for $\gamma'_{x,n}$ and γ'_x. On the other hand, in view of (4.1), for a fixed n one can make $\mathbf{d}_N(\gamma_{x,n}(t), \gamma'_{x,n}(t))$ as small as desired by taking β and β' close enough to each other. $\square$

§5. Appendix 2

We prove here Lemma 1.3. Recall its statement:

Lemma 1.3. *Assume X is a compact manifold endowed with a probability measure μ that is positive on open sets. Let $f \in \mathrm{Diff}^1_\mu(X)$ be a partially hyperbolic diffeomorphism that is dynamically coherent and whose center lamination is plaque expansive. Then the periodic center leaves of f are dense in X.*

For the sake of completeness we recall a few definitions from [HPS], §7.

Let $\mathcal{C}$ be a C^1-lamination of X with leaves of dimension c. A *plaque* is a C^1-embedding $\rho : \mathbb{B}_1 \to X$ of the closed unit ball $\mathbb{B}_1 \subset \mathbb{R}^c$ into one of the leaves of $\mathcal{C}$. A *plaquation* $\mathcal{P}$ is a family $\{\rho\}$ of plaques such that each leaf of $\mathcal{C}$ is the union of the images of the interior of $\mathbb{B}_1$ through some plaques and $\{\rho\}$ is precompact in $\mathrm{Emb}^1(\mathbb{B}_1, X)$. By Theorem 6.2 of [HPS], each C^1-lamination admits a plaquation.

Let f be a diffeomorphism that preserves $\mathcal{C}$. A pseudo orbit $\{x_n\}_{n\in\mathbb{Z}}$ *preserves* $\mathcal{P}$ if for each n, $f(x_n)$ and x_{n+1} lie in a common plaque of $\mathcal{P}$.

The diffeomorphism f is *plaque expansive* if there is an $\varepsilon > 0$ such that if two ε-pseudo orbits $\{p_n\}$ and $\{q_n\}$ both respect $\mathcal{P}$ and satisfy $\mathrm{dist}_X(p_n, q_n) < \varepsilon$ for all n then for each n, p_n and q_n lie in a common plaque of $\mathcal{P}$.

We will use the following Shadowing Lemma ([HPS], Lemma 7A.2 and its proof):

Lemma 5.1. *(Hirsch, Pugh, Shub) Assume that the center distribution of the partially hyperbolic diffeomorphism $f \in \mathrm{Diff}^1(X)$ integrates to a C^1-lamination $\mathcal{C}$ of X. If $(f, \mathcal{C})$ has local product structure and $\nu, \eta > 0$ are given, then there exists $\delta > 0$ such that any δ-pseudo orbit for f can be ν-shadowed by an η-pseudo orbit for f which respects $\mathcal{C}$.* □

Dynamical coherence of f implies that $(f, \mathcal{C})$ has local product structure (see §7A of [HPS] for the definition of the latter).

Proof of Lemma 1.3. Let ε be the constant given by the plaque expansivity of W^c. Pick $0 < \nu < \varepsilon/2$, $\eta = \varepsilon$ and let $0 < \delta$ be given by Lemma 5.1.

Let $B \subset X$ be an open set of diameter at most δ. By the Poincaré Recurrence Theorem, there exists $N \geq 1$ such that $B \cap f^N(B) \neq \emptyset$, hence there is $x \in B$ such that $\mathrm{dist}_X(x, f^N(x)) < \delta$. Consider the N-periodic δ-pseudo orbit

$$\ldots, f^{N-1}(x), x, f(x), f^2(x), \ldots, f^{N-1}(x), x, f(x), f^2(x), \ldots$$

and let $\{y_n\}$ be the η-pseudo orbit that ν-shadows it and respects W^c.

Since the η-pseudo orbits $\{y_n\}$ and $\{z_n\}$, $z_n := y_{n+N}$, satisfy $\mathrm{dist}_X(y_n, z_n) \leq 2\nu$ for all n and both respect W^c, the plaque expansivity of W^c implies that y_n and $z_n = y_{n+N}$ lie in the same leaf of W^c. But $\{y_n\}$ respects W^c, hence $f^N(y_n)$ and y_{n+N} also lie in the same leaf of W^c. Therefore, y_n and $f^N(y_n)$ lie in the same center leaf; we obtain an N-periodic center leaf at a distance at most ν from B. □

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